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Polyakov-Wiegmann formula and multiplicative gerbes

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Abstract: An unambiguous definition of Feynman amplitudes in the Wess-Zumino-Witten sigma model and the Chern-Simon gauge theory with a general Lie group is determined by a certain geometric structure on the group. For the WZW amplitudes, this is a (bundle) gerbe with connection of an appropriate curvature whereas for the CS amplitudes, the gerbe has to be additionally equipped with a multiplicative structure assuring its compatibility with the group multiplication. We show that for simple compact Lie groups the obstruction to the existence of a multiplicative structure is provided by a 2-cocycle of phases that appears in the Polyakov-Wiegmann formula relating the Wess-Zumino action functional of the product of group-valued fields to the sum of the individual contributions. These phases were computed long time ago for all compact simple Lie groups. If they are trivial, then the multiplicative structure exists and is unique up to isomorphism.

Keywords: Chern-Simons Theories, Differential and Algebraic Geometry, Sigma Models

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1 Introduction

It has been known from the early works [1–4] on the Wess-Zumino-Witten (WZW) two-dimensional sigma models that the consistency of such quantum field theories imposes restrictions on the possible values of the coupling constant k called the level. In the more modern geometric language, the consistency requires the existence of a gerbe with connection over the target group G, with the curvature of the gerbe equal to the closed 3-form

$$H_k = \frac{k}{24\pi^2} \mathrm{tr}(g^{-1}dg)^3 \tag{1.1}$$

on G [5–8]. Such a gerbe $\mathcal{G}_k$ exists if and only if the periods of H_k are integers. For simple compact simply-connected groups, this occurs when $k \in \mathbb{Z}$, assuming a proper normalization of the bilinear ad-invariant form $\mathrm{tr}XY$ on the Lie algebra $\mathfrak{g}$ of G that appears on the right hand side of eq. (1.1). For non-simply connected groups, the integrality of the periods of H_k may impose more constraints on the level k. For example, the consistency of the WZW model with the SO(3) target requires even levels. In [4], such restrictions were analyzed for all simple compact groups. Similar results were obtained in [9–12] via an algebraic approach that interpreted the corresponding WZW models as "simple current orbifolds".

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The gerbe $\mathcal{G}_k$ over G determines in a canonical way the "holonomy"

$$\mathcal{H}_{\mathcal{G}_k}(\varphi) \in \mathrm{U}(1)$$

defined for maps φ from a closed oriented surface Σ to G [5, 6, 8]. By definition, such maps are the classical fields of the WZW model and the holonomy $\mathcal{H}_{\mathcal{G}_k}(\varphi)$ defines the contribution of the Wess-Zumino action to the Feynman amplitude of the field φ. The gerbe holonomy is invariant under the composition of fields with orientation-preserving diffeomorphisms D of Σ:

$$\mathcal{H}_{\mathcal{G}_k}(\varphi) = \mathcal{H}_{\mathcal{G}_k}(\varphi \circ D). \tag{1.2}$$

The other important property of the holonomy relating it to the curvature form of the gerbe is the identity

$$\mathcal{H}_{\mathcal{G}_k}(\varphi_1) = \mathcal{H}_{\mathcal{G}_k}(\varphi_0) \exp\left[2\pi i \int\limits_{[0,1]\times\Sigma} \phi^* H_k\right] \tag{1.3}$$

holding for 1-parameter families (i.e. homotopies) of classical fields $\varphi_t = \phi(t,\cdot)$ with $\phi : [0,1] \times \Sigma \to G$.

As noticed in [13], the 2-dimensional WZW theory with simply-connected target groups is closely related to the 3-dimensional Chern-Simons (CS) gauge theory of the same level k. The existence of the CS theory with a non-simply connected gauge group imposes, however, stronger restrictions on the level [14, 15]. For example, the SO(3) CS theory requires k divisible by 4. The topological origin of the difference between the two restrictions has been explained in [16]. In [17], the cohomological discussion of [16] was lifted to the geometric level by showing that the CS theory with gauge group G requires an additional structure on the gerbe $\mathcal{G}_k$ turning it into a "multiplicative gerbe". The argument was completed in [18] by including connections into the discussion of multiplicative structures. It was shown there that a multiplicative gerbe $\mathcal{G}_k$ with connection permits to define unambiguously Feynman amplitudes of the CS theory. More exactly, for every gauge connection A on a G-bundle over a manifold M, it determines canonically a 2-gerbe $\mathcal{K}(A)$ over M (a geometric structure of one degree higher) with curvature equal to the Pontryagin 4-form $\frac{k}{8\pi^2}\mathrm{tr}F(A)^2$. Given a map ϕ of a closed oriented 3-dimensional manifold into M, the CS Feynman amplitude of the gauge field A is given as the holonomy of the 2-gerbe $\mathcal{K}(A)$ along ϕ [18]. It was also shown in [18] that the multiplicative gerbe $\mathcal{G}_k$ determines canonically a central extension of the loop group LG. The latter provides the extended chiral algebra of the corresponding WZW theory whereas the WZW models corresponding to gerbes $\mathcal{G}_k$ without multiplicative structure possess less extended or unextended chiral algebras. E.g. the chiral algebra of the SO(3) WZW theory with k divisible by 4 is the provided by the central extension of $LSO(3)$ whereas for k even but not divisible by 4 it is given by the central extension of the loop group $LSU(2)$ [12].

For simple compact simply-connected groups G, the multiplicative structure on $\mathcal{G}_k$ always exists and is unique up to isomorphism [18]. In the present paper, we address the question of obstructions to the existence of a multiplicative structure on the gerbe $\mathcal{G}_k$ over

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simple compact non-simply connected groups G with fundamental group $\pi_1(G) = Z$, as well as the classification of such structures.

We show that the unique obstruction is provided by the U(1)-valued phases c_{φ_1,φ_2} that appear in the formula

$$\mathcal{H}_{\mathcal{G}_k}(\varphi_1\varphi_2) = c_{\varphi_1,\varphi_2} \cdot \mathcal{H}_{\mathcal{G}_k}(\varphi_1) \cdot \mathcal{H}_{\mathcal{G}_k}(\varphi_2) \cdot \ \mathrm{e}^{\,2\pi i \int\limits_{\Sigma} (\varphi_1 \times \varphi_2)^* \omega_k} \tag{1.4}$$

that relates the holonomy of the point-wise product of two group-valued fields $\varphi_{1,2} : \Sigma \to G$ to the product of the individual holonomies. Above,

$$\omega_k = \frac{k}{8\pi^2} \mathrm{tr}(g_1^{-1} dg_1)(g_2 dg_2^{-1}) \tag{1.5}$$

is a 2-form on the double group $G \times G \equiv G^2$. That eq. (1.4) holds with $c_{\varphi_1,\varphi_2} \equiv 1$ for simply connected groups G is the content of the Polyakov-Wiegmann formula that for the first time appeared (in an equivalent form) in [19], see also [20]. Its generalization to the non-simply-connected groups G was obtained in [4] where the phases c_{φ_1,φ_2} were computed for the surface Σ of genus 1. In the latter case, they reduce to a certain U(1)-valued 2-cocycle c on the group $Z \times Z \equiv Z^2$ that we shall call, accordingly, the FGK cocycle. Our main result states that a multiplicative structure on the gerbe $\mathcal{G}_k$ over a non-simply-connected group G exists if and only if the FGK cocycle is identically equal to 1. Under this condition, such a structure on $\mathcal{G}_k$ is unique up to isomorphism. Thus, the computation of the FGK cocycle for all simple compact Lie groups in [4] provides a complete classification of multiplicative structures on the gerbes $\mathcal{G}_k$.

The paper is organized as follows. In section 2, we recall that a multiplicative structure on the gerbe $\mathcal{G}_k$ requires, in particular, that a certain gerbe with vanishing curvature (i.e. flat) over the group G^2, constructed from the gerbe $\mathcal{G}_k$ over G, be trivial. Isomorphism classes of flat gerbes over G^2 may be identified [6] with cohomology classes of U(1)-valued 2-cocycles on the group G^2. Consequently, such classes provide cohomological obstructions to the triviality of flat gerbes. The corresponding cohomology group is calculated by standard tools of homological algebra. On the other hand, a flat gerbe over G^2 is trivial if and only if its holonomy is trivial. For the flat gerbe mentioned above, the latter property is equivalent to the strict Polyakov-Wiegmann formula without additional phases that may appear in the general case (1.4). We explain in section 3 how such phases give rise to the FGK 2-cocycle. In section 4, we recall from [4] the calculation of this cocycle and in section 5, we clarify the relation between the cohomological obstruction classes and the FGK cocycles by connecting both to bihomomorphisms in $\mathrm{Hom}(Z \otimes Z, \mathrm{U}(1))$. Such bihomomorphisms appeared in the algebraic approach [12, 21] to simple current orbifolds of the WZW models.

The following sections of the paper are devoted to a more thorough discussion of multiplicative gerbes. In section 6, after some preparations, we formulate an abstract definition of a multiplicative gerbe equivariant with respect to the action of a discrete group. This is done in a way that allows to view multiplicative gerbes over non-simply-connected groups G as multiplicative gerbes over their universal covers $\tilde{G}$ that are equivariant under the deck action of $Z = \pi_1(G)$. Section 7 describes equivariant multiplicative gerbes in terms of local data. The local description permits an analysis of obstructions to the existence of equivariant multiplicative gerbes that we perform in section 8. We show that in the case of multiplicative gerbes over the group $\tilde{G}$ equivariant under the deck action of Z, the only obstructions that may be non-trivial belong to the cohomology groups $H^3(Z, \mathrm{U}(1))$ and $H^2(Z^2, \mathrm{U}(1))$. The first one obstructs the existence of the gerbe $\mathcal{G}_k$ over the group $G = \tilde{G}/Z$ and was studied in detail in [22]. The second one in $H^2(Z^2, \mathrm{U}(1))$ is the cohomological obstruction, mentioned above, to the existence of a multiplicative structure on the gerbe $\mathcal{G}_k$. Its triviality is equivalent to the triviality of the FGK 2-cocycle. Finally, in section 9, we discuss equivalences of equivariant multiplicative gerbes and prove that all multiplicative structures on the fixed gerbe $\mathcal{G}_k$ are isomorphic. Conclusions summarize the results of the paper and discuss perspectives for the further work.

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2 Cohomological obstructions

The principal geometric objects that we shall deal with in this paper are hermitian bundle gerbes with unitary connection over a manifold M [23, 24], called below "gerbes" for short. The curvature of a gerbe is a closed 3-form over M. Gerbes over M form a 2-category [25] with objects, 1-morphisms between objects (called also "stable morphisms" or simply "morphisms") and 2-morphisms between 1-morphisms, see also section 2.1 of [26]. One may define tensor product of gerbes, their duals and their pullbacks. The isomorphism classes of flat gerbes form a group that may be naturally identified with the cohomology group $H^2(M, \mathrm{U}(1))$ [6, 8]. We shall study gerbes, equipped with additional structures, over Lie groups.

Let $\tilde{G}$ be a simple, compact, connected and simply-connected Lie group and let Z be a subgroup of its center: $Z \subset Z(\tilde{G})$. The possible cases are $Z = \mathbb{Z}_N$ for some $N \geq 1$ or $Z = \mathbb{Z}_2^2$. The non-cyclic case occurs for $Z = Z(\mathrm{Spin}(4r))$. More complicated discrete Abelian groups appear if one admits non-simple groups $\tilde{G}$ that will not be discussed here. We shall consider the quotient Lie groups $G = \tilde{G}/Z$ that are non-simply connected for non-trivial subgroups Z since $\pi_1(G) = Z$. The deck action of Z on $\tilde{G}$ may be identified with its action by the group multiplication. Eq. (1.1) defines closed bi-invariant 3-forms H_k on G that pull back to 3-forms $\tilde{H}_k$ on $\tilde{G}$ given by the same formula.

Let $\mathcal{G}_k$ be a gerbe with curvature H_k over G. Such a gerbe exists if and only if the 3-form H_k is integral (i.e. has integral 3-periods). The normalization in eq. (1.1) is chosen so that this happens for $k \in \mathbb{Z}$ if the subgroup Z is trivial but only for certain levels $k \in \mathbb{Z}$ for non-trivial Z. Gerbes $\mathcal{G}_k$, when they exist, are unique up to isomorphism except for $G = \mathrm{Spin}(4r)/\mathbb{Z}_2^2 = \mathrm{SO}(4r)/\mathbb{Z}_2$ where, for each $k \in 2\mathbb{Z}$ if r is odd and for each $k \in \mathbb{Z}$ if r is even, there are two isomorphism classes of gerbes with curvature H_k [22]. As already mentioned

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in the Introduction, gerbes $\mathcal{G}_k$ are employed in the definition of Feynman amplitudes in the Wess-Zumino-Witten (WZW) sigma models with target groups G, see also [6–8].

We shall use the notion of a multiplicative gerbe over Lie groups in the version that appeared in [18] as a refinement of the concept introduced in [17]. A multiplicative gerbe may be viewed as an ordinary gerbe over a Lie group equipped with a multiplicative structure. The latter assures the compatibility of the gerbe with the group multiplication. As explained in [17] and [18], the gerbe $\mathcal{G}_k$ over group G equipped with a multiplicative structure canonically determines Feynman amplitudes in the CS theory with gauge group G. A precise definition of a multiplicative gerbe may be found in section 6 below. Here, we shall only briefly elucidate this notion. First, let us remark that the curvature form H_k of the gerbe $\mathcal{G}_k$ satisfies the identity

$$m^*H_k = p_1^*H_k + p_2^*H_k + d\omega_k, \tag{2.1}$$

where $m : G^2 \to G$ is the group multiplication, $p_{1,2} : G^2 \to G$ are the projections and the 2-form ω_k on G^2 is given by eq. (1.5). Explicitly, the relation (2.1) boils down to equality

$$\mathrm{tr}((g_1g_2)^{-1}d(g_1g_2))^3 = \mathrm{tr}(g_1^{-1}dg_1)^3 + \mathrm{tr}(g_2^{-1}dg_2)^3 + 3d\mathrm{tr}(g_1^{-1}dg_1)(g_2dg_2^{-1}) \tag{2.2}$$

that is straightforward to check. A multiplicative structure over the gerbe $\mathcal{G}_k$ realizes a lift of the relation (2.1) from the level of curvature 3-forms to the level of gerbes. More precisely, it involves an isomorphism

$$m^*\mathcal{G}_k \cong p_1^*\mathcal{G}_k \otimes p_2^*\mathcal{G}_k \otimes \mathcal{I}_{\omega_k} \tag{2.3}$$

between gerbes over G^2, where $\mathcal{I}_{\omega_k}$ is a gerbe over G^2 which is trivial except for the global curving 2-form ω_k and the corresponding curvature 3-form $d\omega_k$. The remaining part of the definition of a multiplicative gerbe consists of certain associativity data for the isomorphism (2.3).

The existence of an isomorphism (2.3) is equivalent to the statement that the isomorphism class κ of the flat gerbe

$$m^*\mathcal{G}_k \otimes p_1^*\mathcal{G}_k^* \otimes p_2^*\mathcal{G}_k^* \otimes \mathcal{I}_{-\omega_k} \equiv \mathcal{K}_k \tag{2.4}$$

on G^2 is trivial. Above, $\mathcal{G}_k^*$ denotes the gerbe dual to $\mathcal{G}_k$ with the inverse holonomy and opposite curvature. The isomorphism classes of flat gerbes on G^2 form the cohomology group $H^2(G^2, \mathrm{U}(1))$ so that the isomorphism (2.3) exists if and only if the class $\kappa \in H^2(G^2, \mathrm{U}(1))$ associated to $\mathcal{K}_k$ vanishes. In other words, κ is the cohomological obstruction to the existence of the isomorphism (2.3).

We have to understand the structure of the obstruction cohomology group $H^2(G^2, \mathrm{U}(1))$ for the Lie groups $G = \tilde{G}/Z$. The lowest homology groups (with integer coefficients) for simple, compact, connected and simply-connected Lie group $\tilde{G}$ are

$$H_0(\tilde{G}) = \mathbb{Z}, \qquad H_1(\tilde{G}) = 0 = H_2(\tilde{G}). \tag{2.5}$$

Since Z acts properly on $\tilde{G}$, it follows (see e.g. Corollary 7.3 of [27]) that

$$H_n(G) = H_n(Z) \qquad \text{for} \qquad n = 0, 1, 2, \tag{2.6}$$

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where on the right hand side appear the group-homology groups [27, 28]. One has

$$H_0(Z) = \mathbb{Z}, \qquad H_1(Z) = Z, \qquad H_2(Z) = \begin{cases} 0 & \text{if } Z = \mathbb{Z}_N, \\ \mathbb{Z}_2 & \text{if } Z = \mathbb{Z}_2^2. \end{cases} \tag{2.7}$$

The Universal Coefficients Theorem implies that

$$H^2(G, \mathrm{U}(1)) \cong \mathrm{Hom}(H_2(G), \mathrm{U}(1)) \cong \mathrm{Hom}(H_2(Z), \mathrm{U}(1)) \cong H^2(Z, \mathrm{U}(1)), \tag{2.8}$$

where, in the last member, U(1) is considered as a trivial Z-module. This gives the result

$$H^2(G, \mathrm{U}(1)) \cong \begin{cases} 0 & \text{if } Z = \mathbb{Z}_N, \\ \mathbb{Z}_2 & \text{if } Z = \mathbb{Z}_2^2 \end{cases} \tag{2.9}$$

discussed in detail in [4].

Low homology and cohomology of the product groups $\tilde{G}^2$ and G^2 can be computed similarly. One has:

$$H_0(\tilde{G}^2) = \mathbb{Z}, \qquad H_1(\tilde{G}^2) = 0 = H_2(\tilde{G}^2). \tag{2.10}$$

and

$$H_n(G^2) \cong H_n(Z^2) \qquad \text{for} \qquad n = 0, 1, 2, \tag{2.11}$$

and, from the Universal Coefficients theorem,

$$H^2(G^2, \mathrm{U}(1)) \cong H^2(Z^2, \mathrm{U}(1)), \tag{2.12}$$

where U(1) is considered as a trivial Z^2-module. The groups $H_n(G^2) \cong H_n(Z^2)$ for $n \leq 2$ are easy to compute from the Künneth formula:

$$\begin{aligned} H_0(G^2) &\cong \mathbb{Z}, \\ H_1(G^2) &\cong H_1(G) \oplus H_1(G) \cong Z^2, \qquad (2.13) \\ H_2(G^2) &\cong H_2(G) \oplus H_1(G) \otimes H_1(G) \oplus H_2(G). \qquad (2.14) \end{aligned}$$

More precisely, in the last isomorphism, the injection of the two components $H_2(G)$ into $H_2(G^2)$ is induced by the embeddings $G \ni g \mapsto (g, 1) \in G^2$ and $G \ni g \mapsto (1, g) \in G^2$ whereas the injection of $H_1(G) \otimes H_1(G)$ is given by the cross product. Eqs. (2.6), (2.7) and (2.14) give then the result:

$$H_2(G^2) \cong \begin{cases} \mathbb{Z}_N & \text{if } Z = \mathbb{Z}_N, \\ \mathbb{Z}_2^6 & \text{if } Z = \mathbb{Z}_2^2. \end{cases} \tag{2.15}$$

The Universal Coefficients Theorem implies now that

$$\begin{aligned} H^2(G^2, \mathrm{U}(1)) &\cong \mathrm{Hom}(H_2(G^2), \mathrm{U}(1)) \\ &\cong \mathrm{Hom}(H_2(G) \oplus H_1(G) \otimes H_1(G) \oplus H_2(G), \mathrm{U}(1)) \\ &\cong H^2(G, \mathrm{U}(1)) \oplus \mathrm{Hom}(H_1(G) \otimes H_1(G), \mathrm{U}(1)) \oplus H^2(G, \mathrm{U}(1)) \\ &\cong \begin{cases} \mathbb{Z}_N & \text{if } Z = \mathbb{Z}_N, \\ \mathbb{Z}_2^6 & \text{if } Z = \mathbb{Z}_2^2. \end{cases} \end{aligned}$$

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Eqs. (2.12), (2.6), (2.7) and (2.8) permit to rewrite the latter isomorphisms in terms of the cohomology of finite groups:

$$H^2(Z^2,\mathrm{U}(1)) \cong H^2(Z,\mathrm{U}(1)) \oplus \mathrm{Hom}(Z\otimes Z,\mathrm{U}(1)) \oplus H^2(Z,\mathrm{U}(1)). \tag{2.16}$$

Here, the injections of $H^2(Z,\mathrm{U}(1))$ into $H^2(Z^2,\mathrm{U}(1))$ are induced by considering the U(1)-valued 2-cocycles $c^Z_{z,z'}$ on Z as 2-cocycles on Z^2 according to the formulae

$$c^{Z^2}_{(z_1,z_2),(z'_1,z'_2)} = c^Z_{z_1,z'_1} \qquad \text{or} \qquad c^{Z^2}_{(z_1,z_2),(z'_1,z'_2)} = c^Z_{z_2,z'_2}, \tag{2.17}$$

respectively, whereas the injection of the group $\mathrm{Hom}(Z\otimes Z,\mathrm{U}(1))$ of bihomomorphisms $\zeta : Z^2 \to \mathrm{U}(1)$ into the cohomology group $H^2(Z^2,\mathrm{U}(1))$ may be induced by setting

$$c^{Z^2}_{(z_1,z_2),(z'_1,z'_2)} = \zeta(z_1,z'_2). \tag{2.18}$$

The above information about the cohomology group $H^2(Z^2,\mathrm{U}(1)) \cong H^2(G^2,\mathrm{U}(1))$ will be used in the sequel.

3 Generalized Polyakov-Wiegmann formula and the FGK cocycle

Isomorphic gerbes have the same holonomy. The converse is also true over manifolds for which $H^2(M,\mathrm{U}(1)) = \mathrm{Hom}(H_2(M),\mathrm{U}(1))$ as is the case for groups G or G^2 (this uses the fact that $H_2(M)$ is spanned by images of closed oriented surfaces). The triviality up to isomorphism of the gerbe $\mathcal{K}_k$ of eq. (2.4) is then equivalent to the triviality of its holonomy

$$\begin{aligned}\mathcal{H}_{\mathcal{K}_k}(\varphi_1\times\varphi_2) &= \mathcal{H}_{\mathcal{G}_k}(\varphi_1\varphi_2)\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_1)^{-1}\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_2)^{-1}\cdot\,\mathrm{e}^{-2\pi i\int_\Sigma(\varphi_1\times\varphi_2)^*\omega_k}\\ &= c_{\varphi_1,\varphi_2}\in\mathrm{U}(1)\end{aligned} \tag{3.1}$$

for any pair of maps $\varphi_{1,2} : \Sigma \to G$, see eq. (1.4). Somewhat surprisingly, the generalized Polyakov-Wiegmann formula (1.4) leads to a different picture of obstructions to the existence of an isomorphism (2.3) than the discussion in section 2. Namely, it induces obstructions living in the group of U(1)-valued 2-cocycles on Z^2 and not in the corresponding cohomology group $H^2(Z^2,\mathrm{U}(1)) \cong H^2(G^2,\mathrm{U}(1))$. Such obstruction 2-cocycles may be nontrivial even if their cohomology class is trivial. Here is how this story goes.

1. First, if $\varphi_{1,2,3} : \Sigma \to G$ then the holonomy of $\mathcal{H}_{\mathcal{G}_k}(\varphi_1\varphi_2\varphi_3)$ may be calculated in two different ways. On the one hand,

$$\begin{aligned}\mathcal{H}_{\mathcal{G}_k}(\varphi_1\varphi_2\varphi_3) &= c_{\varphi_1\varphi_2,\varphi_3}\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_1\varphi_2)\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_3)\cdot\,\mathrm{e}^{2\pi i\int_\Sigma(\varphi_1\varphi_2\times\varphi_3)^*\omega_k}\\ &= c_{\varphi_1,\varphi_2}\cdot c_{\varphi_1\varphi_2,\varphi_3}\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_1)\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_2)\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_3)\\ &\qquad\cdot\,\mathrm{e}^{2\pi i\int_\Sigma[(\varphi_1\varphi_2\times\varphi_3)^*\omega_k+(\varphi_1\times\varphi_2)^*\omega_k]}.\end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned}\mathcal{H}_{\mathcal{G}_k}(\varphi_1\varphi_2\varphi_3) &= c_{\varphi_1,\varphi_2\varphi_3}\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_1)\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_2\varphi_3)\cdot\,\mathrm{e}^{2\pi i\int_\Sigma(\varphi_1\times\varphi_2\varphi_3)^*\omega_k}\\ &= c_{\varphi_1,\varphi_2\varphi_3}\cdot c_{\varphi_2,\varphi_3}\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_1)\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_2)\cdot\mathcal{H}_{\mathcal{G}_k}(\varphi_3)\\ &\qquad\cdot\,\mathrm{e}^{2\pi i\int_\Sigma[(\varphi_1\times\varphi_2\varphi_3)^*\omega_k+(\varphi_2\times\varphi_3)^*\omega_k]}.\end{aligned} \tag{3.3}$$

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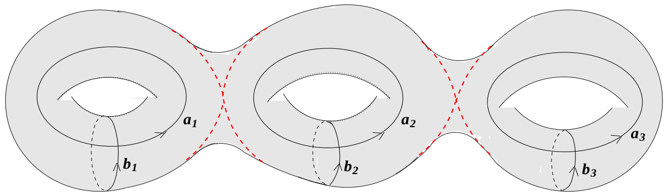

Figure 1. Genus 3 surface with a marking; broken red lines indicate the contours of its version with pinched off handles.

Since a direct calculation shows that on G^3,

$$(m \circ p_{12} \times Id)^* \omega_k + p_{12}^* \omega_k = (Id \times m \circ p_{23})^* \omega_k + p_{23}^* \omega_k \tag{3.4}$$

with the natural notation for the projections $p_{ij} : G^3 \to G^2$, we infer that the exponential terms on the right hand side of eqs. (3.2) and (3.3) coincide and, consequently, that

$$c_{\varphi_1,\varphi_2} \cdot c_{\varphi_1\varphi_2,\varphi_3} = c_{\varphi_1,\varphi_2\varphi_3} \cdot c_{\varphi_2,\varphi_3}, \tag{3.5}$$

i.e. that c_{φ_1,φ_2} is a U(1)-valued 2-cocycle on the group of maps $\varphi : \Sigma \to G$.

2. Second, c_{φ_1,φ_2} depends only on the homotopy classes of φ_1 and φ_2. Indeed, if φ_1 is homotopic to φ_1' and φ_2 is homotopic to φ_2', with $\phi_{1,2} : [0,1] \times \Sigma \to G$ being the corresponding homotopies, then by eq. (1.3),

$$\begin{aligned} \mathcal{H}_{\mathcal{G}_k}(\varphi_1') &= \mathcal{H}_{\mathcal{G}_k}(\varphi_1) \cdot \mathrm{e}^{2\pi i \int\limits_{[0,1]\times\Sigma} \phi_1^* H_k} \\ \mathcal{H}_{\mathcal{G}_k}(\varphi_2') &= \mathcal{H}_{\mathcal{G}_k}(\varphi_2) \cdot \mathrm{e}^{2\pi i \int\limits_{[0,1]\times\Sigma} \phi_2^* H_k} \\ \mathcal{H}_{\mathcal{G}_k}(\varphi_1'\varphi_2') &= \mathcal{H}_{\mathcal{G}_k}(\varphi_1\varphi_2) \cdot \mathrm{e}^{2\pi i \int\limits_{[0,1]\times\Sigma} (\phi_1\phi_2)^* H_k} . \end{aligned}$$

Using the relation (2.1), we infer that

$$c_{\varphi_1',\varphi_2'} = c_{\varphi_1,\varphi_2} \equiv c_{[\varphi_1],[\varphi_2]},$$

where $[\varphi]$ denotes the homotopy class of the map $\varphi : \Sigma \to G$. Such homotopy classes are in one-to-one correspondence with elements of $Z^{2\gamma}$, where γ is the genus of Σ. The element $(z_1, z_2, \ldots, z_{2\gamma-1}, z_{2\gamma})$ corresponding to $[\varphi]$ is given by the holonomies

$$z_{2i-1} = \mathcal{P}\, \mathrm{e}^{\int_{a_i} A_\varphi}, \qquad z_{2i} = \mathcal{P}\, \mathrm{e}^{\int_{b_i} A_\varphi}, \tag{3.6}$$

of the non-Abelian flat gauge field $A_\varphi = \varphi^*(g^{-1}dg)$ on Σ. Above, $\mathcal{P}$ stands for the path-ordering (from left to right) along paths a_i, b_i, $i = 1, \ldots, \gamma$, that generate a fixed marking of Σ, see figure 1. Note that $[\varphi_1\varphi_2] = [\varphi_1][\varphi_2]$, where on the right hand side the product is taken in $Z^{2\gamma}$. We infer that $c_{[\varphi_1],[\varphi_2]}$ is a U(1)-valued 2-cocycle on the finite group $Z^{2\gamma}$.

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3. Third, it is easy to see from the definition of the 2-cocycle $c_{[\varphi_1],[\varphi_2]}$ that if $[\varphi_1] = (z_1, \ldots, z_{2\gamma})$ and $[\varphi_2] = (z'_1, \ldots, z'_{2\gamma})$ then

$$c_{(z_1,\ldots,z_{2\gamma}),(z'_1,\ldots,z'_{2\gamma})} = \prod_{i=1}^{\gamma} c_{(z_{2i-1},z_{2i}),(z'_{2i-1},z'_{2i})}. \tag{3.7}$$

In order to obtain this relation, just calculate $c_{[\varphi_1],[\varphi_2]}$ from the holonomy of the fields defined on a family of surfaces Σ whose handles are pinched away from each other, see figure 1. On the one hand, $c_{[\varphi_1],[\varphi_2]}$ is the same for all surfaces in the family because the holonomies (3.6) are the same. On the other hand, since the fundamental group of G is commutative, the holonomy of A_φ around the pinched curves is trivial and one may take fields that extend smoothly to the limiting surface with pinched handles giving rise the the product expression on the right hand side. It is then enough to consider surface Σ of genus 1, i.e. the torus $\mathbb{T}^2 = S^1 \times S^1$, leading to a U(1)-valued 2-cocycle $c_{(z_1,z_2),(z'_1,z'_2)}$ on the finite group Z^2, the FGK cocycle. Let us stress that it is the non-triviality of the FGK cocycle $c_{(z_1,z_2),(z'_1,z'_2)}$ and *not* of its cohomology class in $H^2(Z^2, \mathrm{U}(1))$, that obstructs the existence of a multiplicative structure on the gerbe $\mathcal{G}_k$ over the group $G = \tilde{G}/Z$.

It is well known that 2-cocycles on a group are related to projective representations. In particular, the FGK 2-cocycle c is related to a projective representation $\Psi \mapsto {}^{(z_1,z_2)}\Psi$ of Z^2 in the space of quantum states of the group $\tilde{G}$ CS theory on the 3-manifold $\mathbb{T}^2 \times \mathbb{R}$ [15]. This space is spanned by the characters of the central extension of the loop group $L\tilde{G}$. The FGK 2-cocycle characterizes the projectivity of the representation:

$${}^{(z_1z'_1,z_2z'_2)}\Psi = c_{(z_1,z_2),(z'_1,z'_2)}\,{}^{(z_1,z_2)}\big({}^{(z'_1,z'_2)}\Psi\big).$$

If $c \equiv 1$ then the representation of Z^2 is genuine rather than projective. In this case, one may define the subspace of the Z^2-invariant states which forms the space of quantum states of the CS theory with the non-simply-connected group $G = \tilde{G}/Z$ on the same manifold $\mathbb{T}^2 \times \mathbb{R}$. The subspace of the Z^2-invariant states Ψ is spanned by the characters of the central extension of the loop group LG that is determined by the corresponding multiplicative gerbe. As already mentioned, such a central extension provides the extended chiral algebra of the corresponding group G WZW model. In particular, when $c \equiv 1$, the toroidal partition function of the group G WZW model is a diagonal combination of the absolute values squared of the characters of the extended chiral algebra.

4 Calculation of the FGK cocycle

The calculation of the 2-cocycles $c_{(z_1,z_2),(z'_1,z'_2)}$ to which case eq. (3.7) reduces the general expression, has been done in ref. [4]. Let us recall (and slightly complete) the argument of [4].

We shall start from the cyclic case when $Z = \mathbb{Z}_N$ with the generator $\zeta = \mathrm{e}^{2\pi i\theta}$ for some $\theta \neq 0$ in the coweight lattice $P^\vee$ of group G. Recall that the existence of a gerbe $\mathcal{G}_k$ with curvature H_k over the group $G = \tilde{G}/Z$ requires the integrality of H_k. As was shown in [4], the latter is equivalent to the condition

$$\tfrac{1}{2}kN\mathrm{tr}\theta^2 \in \mathbb{Z} \tag{4.1}$$

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that selects the admissible levels $k \in \mathbb{Z}$. To each pair (m, n) of integers, we may associate a field configuration $\varphi_{m,n} : \mathbb{T}^2 \to G$ on the two-dimensional torus $\mathbb{T}^2$ such that

$$\varphi_{m,n}(\,\mathrm{e}^{\,i\sigma_1},\,\mathrm{e}^{\,i\sigma_2}) = \mathrm{e}^{\,i(\sigma_1 m\theta + \sigma_2 n\theta)}. \tag{4.2}$$

Note that the homotopy class of $\varphi_{m,n}$

$$[\varphi_{m,n}] = (\zeta^m, \zeta^n) \in Z^2. \tag{4.3}$$

Since $\varphi_{m,n}$ takes values in the circle $\{\,\mathrm{e}^{\,i\sigma\theta} \in G | \sigma \in [0, 2\pi[\}$ and all gerbes over S^1 are trivial up to isomorphism, it follows that

$$\mathcal{H}_{\mathcal{G}_k}(\varphi_{m,n}) = 1. \tag{4.4}$$

As a result,

$$\begin{aligned} c_{(\zeta^m,\zeta^n),(\zeta^{m'},\zeta^{n'})} &= \mathrm{e}^{\,-\frac{ik}{4\pi}\int\limits_{\mathbb{T}^2} \mathrm{tr}\left(\varphi_{mn}^{-1}d\varphi_{mn}\right)\left(\varphi_{m'n'}d\varphi_{m'n'}^{-1}\right)} \\ &= \mathrm{e}^{\,-\frac{ik}{4\pi}\int\limits_0^{2\pi}\int\limits_0^{2\pi} \mathrm{tr}(m\theta d\sigma_1 + n\theta d\sigma_2)(m'\theta d\sigma_1 + n'\theta d\sigma_2)} \\ &= \mathrm{e}^{\,\pi ik(m'n - mn')\mathrm{tr}\theta^2} \equiv c_{(m,n),(m',n')}. \end{aligned} \tag{4.5}$$

Note that the relation (4.1) implies directly that $c_{(m,n),(m',n')}$ depends only on the classes modulo N of the integers m, n, m', n'. Besides, $c_{(m,n),(m',n')} \equiv 1$ if and only if

$$\tfrac{1}{2}k\mathrm{tr}\theta^2 \in \mathbb{Z}.$$

Let us observe that in the particular case when $N = 2$ and $k\mathrm{tr}\theta^2$ is an integer, one has:

$$c_{(m,n),(m',n')} = e_{(m,n)} e^{-1}_{(m+n,m'+n')} e_{(m',n')}$$

for $e_{(m,n)} = \mathrm{e}^{\,\pi ikmn\mathrm{tr}\theta^2}$ so that the FGK 2-cocycle is cohomologically trivial although it is non-trivial if $ktr\theta^2$ is an odd integer as for $Z = Z(\mathrm{SU}(2))$ at even levels not divisible by 4. This shows that the requirement of triviality of the FGK cocycle is, in general, strictly stronger than the requirement of its cohomological triviality.

Consider now the case when $Z = Z(\mathrm{Spin}(4r)) = \mathbb{Z}_2^2$ and is generated by $\zeta_1 = \mathrm{e}^{\,2\pi i\theta_1}$ and $\zeta_2 = \mathrm{e}^{\,2\pi i\theta_2}$ for certain $\theta_1, \theta_2 \in P^\vee$. As was shown in [4], the integrality of the 3-form H_k on $G = \mathrm{Spin}(4r)/\mathbb{Z}_2^2$ imposes now the conditions

$$k\mathrm{tr}\theta_1^2\,, \qquad k\mathrm{tr}\theta_2^2\,, \qquad 2k\mathrm{tr}\theta_1\theta_2 \in \mathbb{Z} \tag{4.6}$$

(ref. [4] considered an additional restriction that required that $k\mathrm{tr}\theta_1\theta_2$ be integral; we drop it here). An inspection shows that $\mathrm{tr}\theta_1\theta_2$ is always a half-integer and that $\mathrm{tr}\theta_1^2$ and $\mathrm{tr}\theta_2^2$ are integers when r is even and, say, the first one is a half-integer and the second one an integer when r is odd. It follows that the gerbes $\mathcal{G}_k$ over $\mathrm{Spin}(4r)/\mathbb{Z}_2^2$ exist for all $k \in \mathbb{Z}$ if r is even and for all $k \in 2\mathbb{Z}$ if r is odd, as already indicated in section 2. Eq. (4.3) with $m\theta$ standing now for $m_1\theta_1 + m_2\theta_2$ and $n\theta$ for $n_1\theta_1 + n_2\theta_2$ associates a field configuration

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$\varphi_{m,n} : \mathbb{T}^2 \to G$ to a pair (m,n) of vectors in $\mathbb{Z}^2$ with $m = (m_1, m_2)$ and $n = (n_1, n_2)$. The relation (4.3) still hold with $\zeta^m \equiv \zeta_1^{m_1}\zeta_2^{m_2}$ and $\zeta^n \equiv \zeta_1^{n_1}\zeta_2^{n_2}$. If the vectors m and n are parallel, then $\varphi_{m,n}$ takes values in a circle in G and, for dimensional reasons, the identity (4.4) still holds. The integral homology group $H_2(G) \cong H_2(Z) \cong \mathbb{Z}_2$ is generated by $\varphi_{(1,0),(0,1)}$ [4]. It is easy to see from the local expression for the gerbe holonomy that

$$\mathcal{H}_{\mathcal{G}_k}(\varphi_{(1,0),(0,1)})^2 = \mathcal{H}_{\mathcal{G}_k}(\varphi_{(2,0),(0,1)}) = (-1)^k. \tag{4.7}$$

The latter equality is obtained by noting that there exists a smooth map $\tilde{g} : D \to G$ defined on the unit disc D such that $\tilde{g}(0) = 1$ and $\tilde{g}(\,\mathrm{e}^{\,i\sigma_1}) = \,\mathrm{e}^{\,2i\sigma_1\theta_1}$ so that $\tilde{g}(t\,\mathrm{e}^{\,i\sigma_1\theta_1})\,\mathrm{e}^{\,i\sigma_2\theta_2}$ provides the homotopy between $\varphi_{(0,0),(0,1)}$ and $\varphi_{(2,0),(0,1)}$. The use of this homotopy leads via eq. (1.3) to the left equality in (4.7) [4]. We infer that

$$\mathcal{H}_{\mathcal{G}_k}(\varphi_{(1,0),(0,1)}) = \pm\,\mathrm{e}^{\,\pi ik/2}.$$

Different choices of the sign correspond to the holonomy of gerbes $\mathcal{G}_k$ in two different isomorphism classes. The invariance (1.2) of the gerbe holonomy under the orientation-preserving diffeomorphisms of $\mathbb{T}^2$ implies that

$$\mathcal{H}_{\mathcal{G}_k}(\varphi_{m,n}) = \mathcal{H}_{\mathcal{G}_k}(\varphi_{\tilde{m},\tilde{n}}) \qquad \text{if} \qquad (\tilde{m},\tilde{n}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(m,n) \tag{4.8}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ (where $m, n, \tilde{m}, \tilde{n}$ are treated as column vectors). Using this invariance, it is easy to see that

$$\mathcal{H}_{\mathcal{G}_k}(\varphi_{m,n}) = \Big(\pm\,\mathrm{e}^{\,\pi ik/2}\Big)^{m\wedge n}$$

whenever the components of m and n are 0 or 1for $m\wedge n = m_1n_2 - m_2n_1$. Employing similar homotopies as before, one may check that this equation remains true for all $m, n \in \mathbb{Z}^2$. Finally, the definition (3.1) gives the result:

$$c_{(\zeta^m,\zeta^n),(\zeta^{m'},\zeta^{n'})} = \Big(\pm\,\mathrm{e}^{\,\pi ik/2}\Big)^{m\wedge n' + m'\wedge n}\ \mathrm{e}^{\,\pi ik\mathrm{tr}(m'\theta n\theta - m\theta n'\theta)} \equiv c_{(m,n),(m',n')}. \tag{4.9}$$

It is straightforward to verify directly using eqs. (4.6) that the right hand side depends on the classes of m, n, m', n' in $\mathbb{Z}^2/2\mathbb{Z}^2$. The $\mathrm{SL}(2,\mathbb{Z})$ symmetry extends to the 2-cocycle $c_{(m,n),(m',n')}$ implying that

$$c_{(m,n),(m',n')} = c_{(\tilde{m},\tilde{n}),(\tilde{m}',\tilde{n}')}$$

if the pairs (m,n) and $(\tilde{m},\tilde{n})$ are related as in (4.8). It is easy to check that $c_{(m,n),(m',n')} \equiv 1$ if and only if the upper sign is chosen on the right hand side of eq. (4.9) and $k \in 2\mathbb{Z}$ if r is even or $k \in 4\mathbb{Z}$ when r is odd. Note that the expression (4.9) for $c_{(m,n),(m',n')}$ encompasses also the formula (4.5) if we set $m \wedge n \equiv 0$ for $Z = \mathbb{Z}_N$.

Summarizing, the obstruction FGK 2-cocycle $c_{(m,n),(m',n')}$ on Z^2 is given by eq. (4.9). We have included a table in section 10 listing those values of k for which the FGK cocycle is trivial.

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5 FGK cocycle and the cohomological obstruction

The obstruction cohomology class $\kappa \in H^2(G^2, \mathrm{U}(1)) \cong \mathrm{Hom}(H_2(G^2), \mathrm{U}(1))$ that corresponds to the isomorphism class of the flat gerbe $\mathcal{K}_k$ over G^2 defined by (2.4) may be easily described explicitly. Indeed, it assigns to fields $\varphi_1 \times \varphi_2 : \Sigma \to G^2$ inducing the homology classes $[\varphi_1 \times \varphi_2] \in H_2(G^2)$ their holonomy with respect to the gerbe $\mathcal{K}_k$:

$$\big\langle [\varphi_1 \times \varphi_2], \kappa \big\rangle = \mathcal{H}_{\mathcal{K}_k}(\varphi_1 \times \varphi_2) = c_{\varphi_1, \varphi_2}, \tag{5.1}$$

see eq. (3.1). In order to describe κ, it is then enough to calculate c_{φ_1,φ_2} for fields $\varphi_{1,2}$ such that $[\varphi_1 \times \varphi_2]$ generate $H_2(G^2)$.

Let us first consider the case with $Z = \mathbb{Z}_N$. Here

$$H_2(G^2) \cong H_1(G) \otimes H_1(G) \cong \mathbb{Z}_N,$$

see (2.14) and (2.15), with the first isomorphism given by the cross product. The group $H_1(G) \cong Z$ is composed of the homology classes of the maps

$$S^1 \ni \mathrm{e}^{\,i\sigma} \longmapsto \mathrm{e}^{\,i\sigma m\theta} \in G$$

that correspond to elements $\zeta^m \in Z$ for $\zeta = \mathrm{e}^{\,2\pi i\theta}$. Consequently, $H_2(G^2)$ is composed of the cross products of the latter classes. These are the homology classes of the maps

$$\mathbb{T}^2 \ni \big(\mathrm{e}^{\,i\sigma_1}, \mathrm{e}^{\,i\sigma_2}\big) \longmapsto \Big(\mathrm{e}^{\,i\sigma_1 m\theta}, \mathrm{e}^{\,i\sigma_2 n\theta}\Big) \in G^2,$$

i.e. of $\varphi_{m,0} \times \varphi_{0,n}$ in the notation of eq. (4.2). Eqs. (5.1) and (4.5) give for the paring of these classes with the cohomology class κ the result:

$$\big\langle [\varphi_{m,0} \times \varphi_{0,n}], \kappa \big\rangle = c_{\varphi_{m,0},\varphi_{0,n}} = \mathrm{e}^{\,-\pi i k m n \mathrm{tr}\theta^2}.$$

The right hand side induces a bihomomorphism $\xi_\kappa : Z^2 \to \mathrm{U}(1)$,

$$\xi_\kappa(\zeta^m, \zeta^n) = \mathrm{e}^{\,-\pi i k m n \mathrm{tr}\theta^2}. \tag{5.2}$$

The discussion of section 2 relating bihomomorphisms to the cohomology classes, see eq. (2.18), permits to identify κ with the class in $H^2(Z^2, \mathrm{U}(1)) \cong H^2(G^2, \mathrm{U}(1))$ generated by the 2-cocycle

$$\chi_{(\zeta^m,\zeta^n),(\zeta^{m'},\zeta^{n'})} = \mathrm{e}^{\,-\pi i k m n' \mathrm{tr}\theta^2}$$

on the group Z^2.

Let us pass now to the case with $Z = \mathbb{Z}_2^2$. Here

$$H_2(G^2) \cong H_2(G) \oplus H_1(G) \otimes H_1(G) \oplus H_2(G) \cong \mathbb{Z}_2^6,$$

see the results (2.14) and (2.15). The first (resp. second) copy of $H_2(G) \cong \mathbb{Z}_2$ injects into $H_2(G^2)$ to the homology classes of the fields $\varphi_{m,n} \times \varphi_{0,0}$ for $m, n \in \mathbb{Z}^2$ (resp. of the fields $\varphi_{0,0} \times \varphi_{m,n}$). The holonomy of gerbe $\mathcal{K}_k$ is trivial along such fields so that

$$\big\langle [\varphi_{m,n} \times \varphi_{0,0}], \kappa \big\rangle = c_{\varphi_{mn},\varphi_{0,0}} = 1 = c_{\varphi_{0,0},\varphi_{m,n}} = \big\langle [\varphi_{0,0} \times \varphi_{m,n}], \kappa \big\rangle.$$

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On the other hand, similarly as before, $H_1(G)\otimes H_1(G)$ injects to the homology classes in $H_2(G^2)$ of the fields $\varphi_{m,0}\times\varphi_{0,n}$ so that

$$\langle[\varphi_{m,0}\times\varphi_{0,n}],\kappa\rangle = c_{\varphi_{m,0},\varphi_{0,n}} = \left(\pm\,\mathrm{e}^{\,\pi ik/2}\right)^{m\wedge n}\ \mathrm{e}^{\,-\pi ik\mathrm{tr}m\theta n\theta},$$

see eq. (4.9). Again, the right hand side induces a bihomomorphism $\xi_\kappa : Z^2\to \mathrm{U}(1)$,

$$\xi_\kappa(\zeta^m,\zeta^n) = \left(\pm\,\mathrm{e}^{\,\pi ik/2}\right)^{m\wedge n}\ \mathrm{e}^{\,-\pi ik\mathrm{tr}m\theta n\theta}. \tag{5.3}$$

It permits to identify κ with the cohomology class in $H^2(Z^2,\mathrm{U}(1))\cong H^2(G^2,\mathrm{U}(1))$ generated by the 2-cocycle

$$\chi_{(\zeta^m,\zeta^n),(\zeta^{m'},\zeta^{n'})} = \left(\pm\,\mathrm{e}^{\,\pi ik/2}\right)^{m\wedge n'}\ \mathrm{e}^{\,-\pi ik\mathrm{tr}m\theta n'\theta}$$

on Z^2, see again eq. (2.18). Note that the formula (5.3) encompasses also the expression (5.2) if, as before, we set $m\wedge n\equiv 0$ for $Z=\mathbb{Z}_N$ and that the bihomomorphisms ξ_κ satisfy the relation

$$\xi_\kappa(\zeta^m,\zeta^m) = \mathrm{e}^{\,-\pi ik\mathrm{tr}(m\theta)^2}. \tag{5.4}$$

It is well known that the elements $z\in Z$ correspond to simple currents J_z of the level k WZW theory [9], i.e. to primary fields that induce under fusion with other primary fields a permutation of the latter. The conformal weights Δ_z of the primary fields J_z satisfy the relation

$$\Delta_{\zeta^m} = \frac{1}{2}k\mathrm{tr}(m\theta)^2 \mathrm{mod}1.$$

The conditions (4.1) for $Z=\mathbb{Z}_N$ or (4.6) for $Z=\mathbb{Z}_2^2$ are equivalent to the requirement that the simple currents J_z for $z\in Z$ be effective (in the terminology of [11]), i.e. that

$$N_z\Delta_z\in\mathbb{Z}\qquad\text{for}\quad z\in Z$$

where N_z stands for the order of the element z. Eq. (5.4) becomes the identity

$$\xi_\kappa(z,z) = \mathrm{e}^{\,-2\pi i\Delta_z}.$$

Bihomomorphisms with the above property on arbitrary groups of effective simple currents have been studied in the context of simple-current orbifolds of conformal field theories in [12]. In [21] they were called the Kreuzer-Schellekens (KS) bihomomorphisms. Note that if

$$\Delta_z\in\mathbb{Z}\qquad\text{for}\quad z\in Z \tag{5.5}$$

then $\xi_\kappa(z,z)=1$. Such bihomomorphisms are called alternating. They are in one-to-one correspondence, see Lemma 3.16 of [21], with the cohomology classes in $H^2(Z,\mathrm{U}(1))$. The latter group is trivial for $Z=\mathbb{Z}_N$ and in this case the condition (5.5) assures the triviality of the KS bihomomorphism ξ_κ. For $Z=\mathbb{Z}_2^2$, however, $H^2(Z,\mathrm{U}(1))=\mathbb{Z}_2$ and even if the condition (5.5) is satisfied, the KS bihomomorphism ξ_κ may be non-trivial which indeed happens for the choice of the gerbe $\mathcal{G}_k$ corresponding to the lower sign on the right hand side of eq. (5.3).

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As we have shown, there is a close relation between the cohomology class κ obstructing the existence of a stable isomorphism (2.3) and the FGK obstruction 2-cocycle c on Z^2 obtained from the generalized Polyakov-Wiegmann formula. The cohomology class κ comes from the KS bihomomorphism $\xi_\kappa : Z^2 \to \mathrm{U}(1)$ of eq. (5.3) via the embedding

$$\mathrm{Hom}(Z \otimes Z, \mathrm{U}(1)) \hookrightarrow H^2(Z^2, \mathrm{U}(1)) \cong H^2(G^2, \mathrm{U}(1)) \tag{5.6}$$

with the first arrow mapping ξ_κ into the cohomology class of the 2-cocycle

$$\chi_{(z_1,z_2),(z_1',z_2')} = \xi_\kappa(z_1, z_2').$$

On the other hand, the FGK 2-cocycle on Z^2 has the form

$$c_{(z_1,z_2),(z_1',z_2')} = \xi_\kappa(z_1, z_2')\xi_\kappa(z_2, z_1')^{-1}.$$

The triviality of the obstruction cohomology class κ generated by χ_κ must be equivalent to the triviality of the KS bihomomorphism ξ_κ. This may be also checked by a direct calculation. On the other hand, the triviality of the bihomomorphism ξ_κ is clearly equivalent to that of the FGK 2-cocycle c. This establishes the equivalence of three different presentations of the obstruction. Note, for example, that in the case with $Z = \mathbb{Z}_2$, the bihomomorphism ξ_κ given by eq. (5.2) is non-trivial if $k\mathrm{tr}\theta^2$ is an odd integer and the corresponding 2-cocycle χ is cohomologically non-trivial whereas, as discussed above, the FGK 2-cocycle c is non-trivial but cohomologically trivial.

6 Equivariant multiplicative gerbes

In this section we shall define multiplicative and equivariant-multiplicative structures on gerbes over Lie groups G. Some preliminary notations will be needed. First we recall that the sequence $\{G^p\}$ of powers of G forms a simplicial manifold. Here we only need one aspect of this assertion, namely that there are "face maps" $\Delta_k^p : G^p \to G^{p-1}$ for all $p > 1$ and $0 \le k \le p$, namely

$$\Delta_k^p(g_1, \ldots, g_p) := \begin{cases} (g_2, \ldots, g_p) & \text{for } k = 0 \\ (g_1, \ldots, g_{k-1}g_k, \ldots, g_p) & \text{for } 1 \le k < p \\ (g_1, \ldots, g_{p-1}) & \text{for } k = p, \end{cases}$$

and that these face maps satisfy the simplicial relations

$$\Delta_h^{p-1} \circ \Delta_k^p = \Delta_{k-1}^{p-1} \circ \Delta_h^p \tag{6.1}$$

for all $h < k$. Such a structure is also called an "incomplete" simplicial manifold. Notice that the group multiplication $m : G \times G \to G$ and the projections $p_{1,2} : G \times G \to G$ can be rediscovered as $\Delta_1^2 = m$, $\Delta_2^2 = p_1$ and $\Delta_0^2 = p_2$. We will sometimes suppress the upper index of Δ_k^p. A differential form $\omega \in \Lambda^n(G^2)$ will be called multiplicative, if

$$\Delta_0^*\omega + \Delta_2^*\omega = \Delta_3^*\omega + \Delta_1^*\omega. \tag{6.2}$$

In this case we denote the n-form (6.2) by ω_Δ.

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Multiplicative structures are considered for pairs $(\mathcal{G},\omega)$ composed of a gerbe $\mathcal{G}$ over G with curvature H and a multiplicative 2-form $\omega \in \Lambda^2(G^2)$ satisfying

$$m^*H = p_1^*H + p_2^*H + \mathrm{d}\omega. \tag{6.3}$$

Our main example will involve the pair $(\mathcal{G}_k,\omega_k)$ composed of a gerbe with the curvature 3-form H_k of eq. (1.1) and of the 2-form ω_k of eq. (1.5). In this case, the identity (6.3) is shown by eq. (2.1) and ω_k is multiplicative due to eq. (3.4). A multiplicative structure [18] on $(\mathcal{G},\omega)$ is a 1-isomorphism

$$\mathcal{M} : m^*\mathcal{G} \to p_1^*\mathcal{G} \otimes p_2^*\mathcal{G} \otimes \mathcal{I}_\omega \tag{6.4}$$

of gerbes over G^2 and a 2-isomorphism

$$\alpha : (\mathrm{id} \otimes \Delta_0^*\mathcal{M} \otimes \mathrm{id}) \circ \Delta_2^*\mathcal{M} \Rightarrow (\Delta_3^*\mathcal{M} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_1\mathcal{M} \tag{6.5}$$

between 1-isomorphisms of gerbes over G^3 which satisfies a natural pentagon axiom over G^4. The condition that ω is multiplicative is required for the existence of α. In the particular case of the pair $(\mathcal{G}_k,\omega_k)$, the isomorphism $\mathcal{M}$ is the isomorphism (2.3) in section 2. Two multiplicative gerbes $(\mathcal{G}^a,\mathcal{M}^a,\alpha^a)$ and $(\mathcal{G}^b,\mathcal{M}^b,\alpha^b)$ are equivalent, if there exists an isomorphism $\mathcal{B} : \mathcal{G}^a \to \mathcal{G}^b$ and a 2-isomorphism

$$\beta : (p_1^*\mathcal{B} \otimes p_2^*\mathcal{B} \otimes \mathrm{id}) \circ \mathcal{M}^a \Rightarrow \mathcal{M}^b \circ m^*\mathcal{B} \tag{6.6}$$

which is compatible with α^a and α^b in a certain way [18]. Equivalent multiplicative gerbes have the same curvature 3-form H and the same multiplicative 2-form ω.

Next we combine a multiplicative structure on a gerbe $\mathcal{G}$ with an equivariant structure. In general, if a discrete group Z acts smoothly on the left on a manifold M over which a gerbe $\mathcal{G}$ is defined, a Z-equivariant structure on $\mathcal{G}$ [26] consists of a collection of isomorphisms

$$\mathcal{A}_z : \mathcal{G} \to z\mathcal{G},$$

where $z\mathcal{G} := (z^{-1})^*\mathcal{G}$, and a collection of 2-isomorphisms

$$\varphi_{z_1,z_2} : z_1\mathcal{A}_{z_2} \circ \mathcal{A}_{z_1} \Rightarrow \mathcal{A}_{z_1z_2}$$

such that the diagram

$$\begin{array}{ccc} z_1z_2\mathcal{A}_{z_3} \circ z_1\mathcal{A}_{z_2} \circ \mathcal{A}_{z_1} & \xRightarrow{\mathrm{id}\circ\varphi_{z_1,z_2}} & z_1z_2\mathcal{A}_{z_3} \circ \mathcal{A}_{z_1z_2} \\ {\scriptstyle z_1\varphi_{z_2,z_3}\circ\mathrm{id}}\Downarrow & & \Downarrow{\scriptstyle \varphi_{z_1z_2,z_3}} \\ z_1\mathcal{A}_{z_2z_3} \circ \mathcal{A}_{z_1} & \xRightarrow[\varphi_{z_1,z_2z_3}]{} & \mathcal{A}_{z_1z_2z_3} \end{array} \tag{6.7}$$

of 2-isomorphisms is commutative. We need the following facts:

1. Suppose that we have two manifolds M_1 and M_2 with smooth left actions of discrete groups Z_1 and Z_2, respectively. Suppose further that $\varphi : Z \to Z'$ is a group homomorphism and that $f : M_1 \to M_2$ is a smooth map that exchanges the actions in the sense that

$$f(zx) = \varphi(z)f(x) \tag{6.8}$$

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for all $x \in M_1$ and $z \in Z_1$. Then, the pullback $f^*\mathcal{G}$ of a Z_2-equivariant bundle gerbe over M_2 carries a canonical Z_1-equivariant structure.

2. In the case when Z acts freely and properly on M, the quotient M/Z is a smooth manifold and the projection $p : M \to M/Z$ is a surjective submersion. The pullback $p^*\mathcal{G}$ of a gerbe $\mathcal{G}$ over M/Z carries a canonical Z-equivariant structure. Conversely, every gerbe $\mathcal{G}$ over M with a Z-equivariant structure defines a "descent" gerbe $\mathrm{Des}_Z(\mathcal{G})$ over M/Z. These two procedures are inverse to each other in an appropriate sense, see [26].

3. Suppose that we have two smooth manifolds M_1 and M_2, both with free and proper left actions of discrete groups Z_1 and Z_2, respectively. Given a group homomorphism $\varphi : Z_1 \to Z_2$ and a smooth map $f : M_1 \to M_2$ satisfying (6.8), there exists a unique map $g : M_1/Z_1 \to M_2/Z_2$ between the quotients such that $p_2 \circ f = g \circ p_1$. Then,

$$\mathrm{Des}_{Z_1} \circ f^* = g^* \circ \mathrm{Des}_{Z_2}. \tag{6.9}$$

   Thus, descent is compatible with pullbacks. It is also compatible with tensor products.

We further need the definition of Z-equivariant isomorphisms and 2-isomorphisms. For an isomorphism being Z-equivariant is not a property but additional structure. A Z-equivariant structure on an isomorphism $\mathcal{B} : \mathcal{G}^a \to \mathcal{G}^b$ between gerbes with Z-equivariant structures $(\mathcal{A}^a_z, \varphi^a_{z_1,z_2})$ and $(\mathcal{A}^b_z, \varphi^b_{z_1,z_2})$ is a 2-isomorphism

$$\eta_z : z\mathcal{B} \circ \mathcal{A}^a_z \Rightarrow \mathcal{A}^b_z \circ \mathcal{B}$$

such that the diagram

$$\begin{array}{ccc}
z_1z_2\mathcal{B} \circ z_1\mathcal{A}^a_{z_2} \circ \mathcal{A}^a_{z_1} & \xRightarrow{\mathrm{id}_{z_1z_2\mathcal{B}} \circ \varphi^a_{z_1,z_2}} & z_1z_2\mathcal{B} \circ \mathcal{A}^a_{z_1z_2} \\
{\scriptstyle z_1\eta_{z_2} \circ \mathrm{id}_{\mathcal{A}^a_{z_1}}} \Downarrow & & \\
z_1\mathcal{A}^b_{z_2} \circ z_1\mathcal{B} \circ \mathcal{A}^a_{z_1} & & \Downarrow {\scriptstyle \eta_{z_1z_2}} \\
{\scriptstyle \mathrm{id}_{z_1\mathcal{A}^b_{z_2}} \circ \eta_{z_1}} \Downarrow & & \\
z_1\mathcal{A}^b_{z_2} \circ \mathcal{A}^b_{z_1} \circ \mathcal{B} & \xRightarrow[\varphi^b_{z_1,z_2} \circ \mathrm{id}_{\mathcal{B}}]{} & \mathcal{A}^b_{z_1z_2} \circ \mathcal{B}
\end{array} \tag{6.10}$$

of 2-isomorphisms is commutative. Finally, a Z-equivariant 2-isomorphism $\phi : \mathcal{B} \Rightarrow \mathcal{B}'$ between isomorphisms $\mathcal{B}$ and $\mathcal{B}'$ with Z-equivariant structures η_z and η'_z, respectively, is called Z-equivariant, if the diagram

$$\begin{array}{ccc}
z\mathcal{B} \circ \mathcal{A}^a_z & \xRightarrow{\eta_z} & \mathcal{A}^b_z \circ \mathcal{B} \\
{\scriptstyle z\phi \circ \mathrm{id}_{\mathcal{A}^a_z}} \Downarrow & & \Downarrow {\scriptstyle \mathrm{id}_{\mathcal{A}^b_z} \circ \phi} \\
z\mathcal{B}' \circ \mathcal{A}^a_z & \xRightarrow[\eta'_z]{} & \mathcal{A}^b_z \circ \mathcal{B}'
\end{array} \tag{6.11}$$

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of 2-isomorphisms is commutative. In case of a free and proper group action, equivariant isomorphisms and 2-isomorphisms descent to the quotient in a way compatible with pullbacks and tensor products.

In order to combine a multiplicative structure with an equivariant structure, the action $\rho : Z \times G \to G$ has to be compatible with the group multiplication of G in the sense that ρ is a group homomorphism. We will call such group actions "multiplicative". Multiplicative groups actions have the following two properties: if we let Z^p act component-wise on G^p, the face maps $\Delta_k^p : G^{p+1} \to G^p$ introduced above satisfy condition (6.8), where the group homomorphism $\varphi : Z^p \to Z^{p-1}$ is given by the face map Δ_k^p of the group Z. In other words, there are commutative diagrams

$$\begin{array}{ccc} G^p & \xrightarrow{\;z\;} & G^p \\ {\scriptstyle\Delta_k^p}\downarrow & & \downarrow{\scriptstyle\Delta_k^p} \\ G^{p-1} & \xrightarrow[\Delta_k^p(z)]{} & G^{p-1} \end{array} \tag{6.12}$$

for all p and $0 \leq k \leq p$. This property of the action ρ guarantees, for instance, that if $\mathcal{G}$ is a Z-equivariant gerbe over G, the pullbacks $m^*\mathcal{G}$, $p_1^*\mathcal{G}$ and $p_2^*\mathcal{G}$ are Z^2-equivariant gerbes over G^2. The second property of a multiplicative group action is that in case of a free and proper group action, in which the quotient G/Z is again a Lie group, the projection $p : G \to G/Z$ is a Lie group homomorphism. Most importantly, all of this holds for Z a subgroup of the center of G acting by multiplication.

Given a multiplicative group action, equivariant multiplicative structures are considered for pairs $(\mathcal{G}, \omega)$ of a gerbe $\mathcal{G}$ over G and a multiplicative, Z^2-invariant 2-form $\omega \in \Lambda^2(G^2)$ satisfying (6.3) as before. Notice that such 2-forms define Z^2-equivariant trivial bundle gerbes $\mathcal{I}_\omega$. We say that a Z-multiplicative structure on $(\mathcal{G}, \omega)$ is a Z-equivariant structure $(\mathcal{A}_z, \varphi_{z_1,z_2})$ on $\mathcal{G}$, a Z^2-equivariant isomorphism $(\mathcal{M}, \eta_{z_1,z_2})$ like in (6.4) and a Z^3-equivariant 2-isomorphism α like in (6.5), satisfying the pentagon axiom. Two Z-multiplicative gerbes are equivalent, if there exists a Z-equivariant isomorphism $(\mathcal{B}, \kappa_z) : \mathcal{G}^a \to \mathcal{G}^b$ and a Z^2-equivariant 2-isomorphism β like in (6.6), satisfying the same compatibility condition.

The purpose of Z-multiplicative gerbes over G is that they correspond, for a free and proper group action, to multiplicative gerbes over the quotient $G' = G/Z$. This follows from the properties of equivariant structures listed above: the Z-equivariant gerbe $\mathcal{G}$ determines a bundle gerbe $\mathcal{G}' := \mathrm{Des}(\mathcal{G})$ over G'. Eq. (6.9) implies that $m_{12}^*\mathcal{G}' = \mathrm{Des}(m^*\mathcal{G})$, and similarly, $p_i^*\mathcal{G}' = \mathrm{Des}(p_i^*\mathcal{G})$ for $i = 1, 2$. Further, the Z^2-equivariant, multiplicative 2-form ω determines a 2-form $\omega' \in \Lambda^2(G'^2)$, and this 2-form is again multiplicative. Thus, the Z^2-equivariant 1-isomorphism $\mathcal{M}$ determines a 1-isomorphism

$$\mathcal{M}' := \mathrm{Des}(\mathcal{M}) : m^*\mathcal{G}' \to p_1^*\mathcal{G}' \otimes p_2^*\mathcal{G}' \otimes \mathcal{I}_{\omega'}.$$

In the same way, the Z^3-equivariant 2-isomorphism α determines a 2-isomorphism α' as required for a multiplicative gerbe over G'. This 2-isomorphism α' automatically satisfies the pentagon axiom. Thus every Z-multiplicative gerbe over G determines a multiplicative gerbe over the the quotient G'. In the same way, equivalent Z-multiplicative gerbes determine equivalent multiplicative gerbes over G'.

Summarizing, if Z is a discrete group acting on the left on a Lie group G in a smooth, multiplicative and free and proper way, we have a bijection between equivalence classes of Z-multiplicative gerbes over G and equivalence classes of multiplicative gerbes over G/Z. The goal of the following sections is to classify Z-multiplicative structures on the pairs $(\mathcal{G}_k, \omega_k)$ over all compact, simple and simply-connected Lie groups G, for Z a subgroup of the center of G.

7 Local description of equivariant multiplicative gerbes

In this section we connect the geometrical definition of equivariant multiplicative gerbes to the cohomological language used in the first five sections. The cohomology theory that is most appropriate for gerbes, i.e. hermitian bundle gerbes with unitary connection, is the (real) Deligne cohomology. We shall recall some basic facts about it [6, 26, 29].

Let us first consider a general manifold M. We denote by $\mathcal{U}$ the sheaf of smooth U(1)-valued functions, and by Λ^q the sheaf of q-forms. For $\mathfrak{O}$ an open cover of M, the Deligne cohomology $\mathbb{H}^n(\mathfrak{O}, \mathcal{D}(2))$ is the cohomology of the complex

$$0 \longrightarrow A^0(\mathfrak{O}) \xrightarrow{D_0} A^1(\mathfrak{O}) \xrightarrow{D_1} A^2(\mathfrak{O}) \xrightarrow{D_2} A^3(\mathfrak{O}) \longrightarrow \cdots$$

with the cochain groups

$$\begin{aligned} A^0(\mathfrak{O}) &= C^0(\mathfrak{O},\mathcal{U}), \\ A^1(\mathfrak{O}) &= C^0(\mathfrak{O},\Lambda^1) \oplus C^1(\mathfrak{O},\mathcal{U}), \\ A^2(\mathfrak{O}) &= C^0(\mathfrak{O},\Lambda^2) \oplus C^1(\mathfrak{O},\Lambda^1) \oplus C^2(\mathfrak{O},\mathcal{U}), \\ A^3(\mathfrak{O}) &= C^1(\mathfrak{O},\Lambda^2) \oplus C^2(\mathfrak{O},\Lambda^1) \oplus C^3(\mathfrak{O},\mathcal{U}). \end{aligned}$$

Here, $C^\ell(\mathfrak{O}, \mathcal{S})$ denotes the ℓ^{th} Čech cochain group of the open cover $\mathfrak{O}$ with values in a sheaf $\mathcal{S}$ of Abelian groups. The differentials are

$$\begin{aligned} D_0(f_i) &= \left(-\mathrm{i} f_i^{-1} df_i\ ,\ f_j^{-1} \cdot f_i\right), \\ D_1(\Pi_i, \chi_{ij}) &= \left(d\Pi_i\ ,\ -\mathrm{i}\chi_{ij}^{-1} d\chi_{ij} + \Pi_j - \Pi_i\ ,\ \chi_{jk}^{-1} \cdot \chi_{ik} \cdot \chi_{ij}^{-1}\right), \\ D_2(B_i, A_{ij}, g_{ijk}) &= \left(dA_{ij} - B_j + B_i\ ,\ -\mathrm{i} g_{ijk}^{-1} dg_{ijk} + A_{jk} - A_{ik} + A_{ij}\ ,\ g_{jkl}^{-1} \cdot g_{ikl} \cdot g_{ijl}^{-1} \cdot g_{ijk}\right). \end{aligned}$$

A refinement $r : \mathfrak{O}' \to \mathfrak{O}$ of open covers induces the restriction maps $\mathbb{H}^n(\mathfrak{O}, \mathcal{D}(2)) \to \mathbb{H}^n(\mathfrak{O}', \mathcal{D}(2))$ turning the Deligne cohomology groups into a direct system of groups. Its direct limit is denoted $\mathbb{H}^n(M, \mathcal{D}(2))$.

Let us briefly recall what local data of gerbes, isomorphisms and 2-isomorphisms are, for the details we refer the reader to [26]. For a given gerbe $\mathcal{G}$ over M, one can choose a sufficiently "good" open cover $\mathfrak{O}$ of M that permits to extract a cocycle $c \in A^2(\mathfrak{O})$, $D_2 c = 0$, in a certain way. Suppose that two gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ are given, and $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are open covers that permit to extract cocycles c_1 and c_2. Suppose further that $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ is an isomorphism. Then one can choose a common refinement $\mathfrak{O}$ of $\mathfrak{O}_1$ and $\mathfrak{O}_2$ that permits to extract a cochain $b \in A^1(\mathfrak{O})$ such that $c_2 = c_1 + D_1 b$. The cochains for

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isomorphisms add under the composition of these isomorphisms. Finally, if a 2-isomorphism $\varphi : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$ is given and b_1 and b_2 are cochains for $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, for a suitable open cover $\mathfrak{O}$, one can always extract a cochain $a \in A^0(\mathfrak{O})$ such that $b_2 = b_1 + D_0 a$. The cochains for 2-isomorphisms add under both the horizontal and the vertical composition of 2-isomorphisms. Conversely, one can reconstruct gerbes, isomorphisms and 2-isomorphisms from given local data, and the two procedures are inverse to each other in an appropriate sense. In particular, they establish a bijection between $\mathbb{H}^2(M, \mathcal{D}(2))$ and the set of isomorphism classes of gerbes over M.

In the following we want to apply the procedure of extraction of local data to an equivariant multiplicative gerbe. This requires a careful discussion of open covers $\mathfrak{O}^p = \{O_i^p\}_{i \in I^p}$ of powers G^p of a Lie group G. As the definition of a Z-multiplicative gerbe over G involves pullbacks along the face maps $\Delta_k^p : G^p \to G^{p-1}$ and along the action $\rho : Z \times G \to G$, we need the open covers to be compatible with all these maps.

We say that a sequence $\{\mathfrak{O}^p\}$ of open covers of G^p is "simplicial", if the sequence $\{\mathcal{I}^p\}$ of index sets forms an incomplete simplicial set (i.e. there are face maps $\Delta_k^p : I^p \to I^{p-1}$ satisfying (6.1)), such that

$$\Delta_k^p(O_i^p) \subset O^{p-1}_{\Delta_k^p(i)} \tag{7.1}$$

for all $p > 1$, all $0 \leq k \leq p$ and all $i \in I^p$. For a simplicial sequence of open covers one has induced chain maps

$$\left(\Delta_k^p\right)^* : C^\ell(\mathfrak{O}^{p-1}, \mathcal{S}) \to C^\ell(\mathfrak{O}^p, \mathcal{S}) \quad \text{defined by} \quad \left((\Delta_l^p)^* f\right)_i := (\Delta_k^p)^*(f_{\Delta_k^p(i)}), \tag{7.2}$$

satisfying the co-simplicial relations

$$\left(\Delta_k^p\right)^* \circ \left(\Delta_h^{p-1}\right)^* = (\Delta_h^p)^* \circ (\Delta_{k-1}^p)^* \tag{7.3}$$

for $h < k$. We further say that a sequence $\{\mathfrak{O}^p\}$ of open covers is "Z-equivariant", if each cover $\mathfrak{O}^p$ is Z^p-equivariant in the sense that its index set I^p carries an action of Z^p in such a way that

$$z(O_i^p) \subset O_{zi}^p \tag{7.4}$$

for all $z \in Z^p$ and $i \in I^p$. For a Z-equivariant sequence of open covers one has an induced action of Z^p on $C^\ell(\mathfrak{O}^p, \mathcal{S})$ by chain maps, namely

$$z : C^\ell(\mathfrak{O}^p, \mathcal{S}) \to C^\ell(\mathfrak{O}^p, \mathcal{S}) \quad \text{defined by} \quad (zf)_i := (z^{-1})^*(f_{z^{-1}(i)}) \tag{7.5}$$

for $z \in Z^p$ and $f \in C^\ell(\mathfrak{O}^p, \mathcal{S})$. Combining both notions, we say that a sequence $\{\mathfrak{O}^p\}$ of open covers is "Z-simplicial", if it is both simplicial and Z-equivariant, and if the face maps $\Delta_k^p : I^p \to I^{p-1}$ and the action of Z^p on I^p commute in the sense that all diagrams

$$\begin{array}{ccc} I^p & \xrightarrow{\;z\;} & I^p \\ {\scriptstyle \Delta_k^p}\downarrow & & \downarrow{\scriptstyle \Delta_k^p} \\ I^{p-1} & \xrightarrow[\Delta_k^p(z)]{} & I^{p-1} \end{array} \tag{7.6}$$

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are commutative (cf. diagram (6.12)). This compatibility condition ensures that the induced maps (7.2) and (7.5) on the Čech cohomology groups commute in the same way, i.e.

$$z \circ (\Delta_k^p)^* = (\Delta_k^p)^* \circ \Delta_k^p(z). \tag{7.7}$$

In the following we shall use Z-simplicial sequences of open covers to extract local data of Z-multiplicative gerbes. We have included an appendix (after section 10) in which we prove the following. Suppose that a finite Abelian group Z acts on G in a smooth, multiplicative, free and proper way, and assume that $\{\mathfrak{V}^p\}$ is any sequence of open covers $\mathfrak{V}^p$ of G^p. Then, there exists a Z-simplicial sequence $\{\mathfrak{O}^p\}$ of open covers, such that each $\mathfrak{O}^p$ is a refinement of $\mathfrak{V}^p$. As a consequence, one can choose open covers $\mathfrak{V}^p$ separately for all p, in such a way that they permit to extract local data of any given combination of gerbes and isomorphisms. Since each open cover $\mathfrak{O}^p$ of a Z-simplicial refinement is finer than $\mathfrak{V}^p$, also the new covers $\mathfrak{O}^\mathfrak{p}$ permit to extract local data of the given structure. We can hence assume that one can always choose sufficiently fine Z-simplicial sequences of open covers.

For a given Z-simplicial sequence $\{\mathfrak{O}^p\}$ of open covers, we consider the groups

$$K^{p,q,n} := \mathrm{Map}((Z^p)^q, A^n(\mathfrak{O}^p))$$

with elements denoted like $x_{z_1,\ldots,z_q} \in A^n(\mathfrak{O}^p)$, for $z_1, \ldots, z_q$ elements in Z^p. On the groups $K^{p,q,n}$ we find three operators: the first is the Deligne differential

$$D_{p,q,n} : K^{p,q,n} \to K^{p,q,n+1} \quad \text{with} \quad (D_{p,q,n}(x))_{z_1,\ldots,z_q} := D_n(x_{z_1,\ldots,z_q}).$$

The second is the "group cohomology differential of the group Z^p"

$$\delta_{p,q,n}: K^{p,q,n} \to K^{p,q+1,n} \quad \text{with} \quad (\delta_{p,q,n}(x))_{z_0,\ldots,z_q} := z_0 x_{z_1,\ldots,z_q} - x_{z_0 z_1,\ldots,z_q} + \ldots \pm x_{z_0,\ldots,z_{q-1}},$$

whose definition uses the lift (7.5) of the Z^p-action to the Deligne cochain group $A^n(\mathfrak{O}^p)$. The third operator we have is the simplicial operator

$$\Delta_{p,q,n} : K^{p,q,n} \to K^{p+1,q,n} \quad \text{with} \quad (\Delta_{p,q,n}(x))_{z_1,\ldots,z_q} := \sum_{k=0}^{p+1} (-1)^k \left(\Delta_k^{p+1}\right)^* \left(x_{\Delta_k^p z_1,\ldots,\Delta_k^p z_q}\right), \tag{7.8}$$

whose definition uses the lift (7.2) of the face maps to the Deligne cochain groups. Notice that in (7.8) $z_1, \ldots, z_q$ are elements of Z^{p+1} and $\Delta_k^p : Z^{p+1} \to Z^p$ is the face map of the group Z. Due to the co-simplicial relations (7.3), we have $\Delta_{p+1,q,n} \circ \Delta_{p,q,n} = 0$. The Deligne differential D commutes with pullbacks, and thus with both operators δ and Δ. Further, the differentials δ and Δ commute due to (7.7). This endows $K^{p,q,n}$ with the structure of a triple complex.

Now we are prepared to list local data of a Z-equivariant multiplicative gerbe over G. We chose a Z-simplicial sequence $\{\mathfrak{O}^p\}$ of open covers that permit to extract local data of $\mathcal{G}$ and all involved isomorphisms and 2-isomorphisms. Then, the Z-equivariant gerbe $(\mathcal{G}, \mathcal{A}_z, \varphi_{z_1,z_2})$ has local data $c \in K^{1,0,2}$, $b \in K^{1,1,1}$ and $a \in K^{1,2,0}$ satisfying the relations

$$D_{1,0,2}c = 0 \quad , \quad \delta_{1,0,2}c = D_{1,1,1}b \quad , \quad \delta_{1,1,1}b = -D_{1,2,0}a \quad \text{and} \quad \delta_{1,2,0}a = 0,$$

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of which the last one is the commutativity of diagram (6.7), see [26]. The Z^2-equivariant isomorphism $(\mathcal{M}, \eta_{z_1,z_2})$ has local data $\beta \in K^{2,0,1}$ and $\phi \in K^{2,1,0}$ satisfying

$$\Delta_{1,0,2}c + \omega = D_{2,0,1}\beta, \quad \delta_{2,0,1}\beta = \Delta_{1,1,1}b - D_{2,1,0}\phi \quad \text{and} \quad \Delta_{1,2,0}a + \delta_{2,1,0}\phi = 0, \tag{7.9}$$

where the third is the commutativity of diagram (6.10), and the 2-form ω is regarded as an element in $A^2(\mathfrak{O}^2)$ corresponding to the trivial gerbe $\mathcal{I}_\omega$. Finally, the Z^3-equivariant 2-isomorphism α has local data $d \in K^{3,0,0}$ such that

$$\Delta_{2,0,1}\beta = D_{3,0,0}d, \quad \Delta_{2,1,0}\phi + \delta_{3,0,0}d = 0 \quad \text{and} \quad \Delta_{3,0,0}d = 0.$$

where the second is the commutativity of diagram (6.11) and the third is the pentagon axiom for α.

We also need to relate local data of two equivalent Z-multiplicative gerbes. Suppose the equivalence is expressed by a Z-equivariant isomorphism $(\mathcal{B}, \kappa_z)$ and a Z^2-equivariant 2-isomorphism β as discussed in section 6. Suppose further that we have chosen a Z-simplicial sequence of open covers that are fine enough to extract local data of all involved gerbes and isomorphisms. Then, the Z-equivariant isomorphism $(\mathcal{B}, \kappa_z)$ has local data $r \in K^{1,0,1}$ and $s \in K^{1,1,0}$, and the Z^2-equivariant 2-isomorphism β has local data $t \in K^{2,0,0}$. These relate local data $(c_1, b_1, a_1, \beta_1, \phi_1, d_1)$ and $(c_2, b_2, a_2, \beta_2, \phi_2, d_2)$ of the Z-multiplicative gerbes by

$$c_2 = c_1 + D_{1,0,1}r, \quad b_2 = b_1 + \delta_{1,0,1}r + D_{1,1,0}s \quad \text{and} \quad a_2 = a_1 - \delta_{2,0,1}s, \tag{7.10}$$

the last equation expressing the equivariance of $(\mathcal{B}, \kappa_z)$, and

$$\beta_2 = \beta_1 + \Delta_{1,0,1}r + D_{2,0,0}t, \quad \phi_2 = \phi_1 + \Delta_{1,1,0}s - \delta_{2,0,0}t, \quad \text{and} \quad d_2 = d_1 - \Delta_{2,0,0}t. \tag{7.11}$$

We remark that the local data of a Z-multiplicative gerbe does not automatically define a cocycle in the total complex of the triple complex $K^{p,q,n}$, due to the appearance of the 2-form ω in (7.9). In [18] the 2-form has been included into the complex, but here this will not be necessary.

8 Obstructions against equivariant multiplicative structures

Let G be a 2-connected Lie group, and let Z be a finite group acting smoothly and multiplicatively on G. Let $\mathcal{G}$ be a gerbe over G of curvature H, and let $\omega \in \Lambda^2(G^2)$ be a multiplicative 2-form such that identity (6.3) is satisfied.

Due to the 2-connectedness of G, all the elements of a Z-multiplicative structure on $\mathcal{G}$ exist: the isomorphisms $\mathcal{A}_z$ of the equivariant structure on $\mathcal{G}$ and the isomorphism $\mathcal{M}$ over G^2 exist because they are isomorphisms between gerbes of equal curvature, and such gerbes are necessarily isomorphic. The 2-isomorphisms φ_{z_1,z_2} of the equivariant structure on $\mathcal{G}$, the 2-isomorphisms η_{z_1,z_2} of the equivariant structure on $\mathcal{M}$, and the 2-isomorphism α of the multiplicative structure exist because every two isomorphisms between fixed gerbes are necessarily 2-isomorphic over simply connected spaces. *Not* automatically guaranteed are the various conditions these 2-isomorphisms have to satisfy, namely the commutativity of the diagrams (6.7), (6.10), (6.11) and the pentagon axiom for α.

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In order to handle all these conditions, let us extract local data for the structure that we have chosen. We choose an appropriate Z-simplicial sequence $\{\mathfrak{O}^p\}$ as discussed in section 7. According to the discussion there, we obtain local data (c, b, a) for the Z-equivariant gerbe $\mathcal{G}$, local data (β, ϕ) for the Z^2-equivariant isomorphism $\mathcal{M}$ and local data d for the 2-isomorphism α. All relations are satisfied except

$$\delta_{1,2,0}a = 0, \quad \Delta_{1,2,0}a + \delta_{2,1,0}\phi = 0, \quad \Delta_{2,1,0}\phi + \delta_{3,0,0}d = 0 \quad \text{and} \quad \Delta_{3,0,0}d = 0, \tag{8.1}$$

corresponding to the above-mentioned diagrams and the pentagon axiom, respectively. These four equations are the cocycle condition for the 3-cochain (a, ϕ, d) in the total complex of the double complex $K^{p,q,0}$ with differentials $\Delta_{p,q,0}$ and $\delta_{p,q,0}$. In the following we suppress the third index 0.

In the generic case our chosen data does not satisfy the conditions (8.1), and we define

$$u_0 := \delta_{1,2}a \in K^{1,3} \qquad u_1 := \Delta_{1,2}a + \delta_{1,2}\phi \in K^{2,2}$$
$$u_2 := \Delta_{2,1}\phi - \delta_{3,0}d \in K^{3,1} \qquad u_3 := \Delta_{3,0}d \in K^{4,0}.$$

Since this is the coboundary of (a, ϕ, d), the 4-cochain $u := (u_0, u_1, u_2, u_3)$ is a 4-cocycle. By construction, $D_0 u_i = 0$ for all $i = 0, \ldots, 3$. Now we recall the identification

$$\ker D_0|_{A^0(\mathfrak{O}^p)} \cong \mathrm{U}(1) \tag{8.2}$$

due to the fact that all manifolds G^p are connected. We have thus induced identifications

$$K^{p,q,0} \supset \ker D_{p,q,0} \cong C^q(Z^p, \mathrm{U}(1)), \tag{8.3}$$

where $C^q(Z^p, \mathrm{U}(1))$ is the q-th cochain group of the group Z^p with coefficients in $\mathrm{U}(1)$, with Z^p acting trivially on the coefficients. Unlike in section 1–5, we shall use below the additive notation rather than the multiplicative one for the $\mathrm{U}(1)$-valued cochains. Under the identification (8.3), the remaining differentials $\delta_{p,q}$ and $\Delta_{p,q}$ are

$$\delta_{p,q} : C^q(Z^p, \mathrm{U}(1)) \to C^{q+1}(Z^p, \mathrm{U}(1)) \quad \text{and} \quad \Delta_{p,q} : C^q(Z^p, \mathrm{U}(1)) \to C^q(Z^{p+1}, \mathrm{U}(1)),$$

with $\delta_{p,q}$ the usual group cohomology differential for the group Z^p. The following identities are straightforward to check:

1. $\delta_{p,0} = 0$, since the action of Z^p on the coefficients $\mathrm{U}(1)$ is trivial,
2. $\Delta_{p,0} : \mathrm{U}(1) \to \mathrm{U}(1) : z \mapsto \begin{cases} 1 & \text{for } p \text{ even}, \\ z & \text{for } p \text{ odd}, \end{cases}$
3. $\Delta_{p,1} = \delta_{1,p}$ under the set-theoretic equality $C^1(Z^p, \mathrm{U}(1)) = C^p(Z, \mathrm{U}(1))$,
4. $\Delta_{p,p}u = 0$ if and only if $\delta_{p,p}u = 0$.

The total cohomology of the double complex $C^q(Z^p, \mathrm{U}(1))$ is denoted $\mathbb{H}^n_0(Z, \mathrm{U}(1))$, where the 0 indicates that our double complex starts at $q = 0$ (but at $p = 1$). The cocycle

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$u := (u_0, \ldots, u_3)$ we have obtained from the pair $(\mathcal{G}, \omega)$ as described above is now a 4-cocycle and defines a class $[u] \in \mathbb{H}^4_0(Z, \mathrm{U}(1))$. The cocycle conditions are

$$\delta_{1,3}u_0 = 0 \quad , \quad \Delta_{1,3}u_0 = \delta_{2,2}u_1 \quad , \quad \Delta_{2,2}u_1 = \delta_{3,1}u_2 \quad \text{and} \quad \Delta_{3,1}u_2 = 0. \tag{8.4}$$

Before we proceed investigating more closely the class $[u] \in \mathbb{H}^4_0(Z, \mathrm{U}(1))$, let us make two general claims about it. The first is that $[u]$ is uniquely determined by the pair $(\mathcal{G}, \omega)$ of the gerbe $\mathcal{G}$ and the 2-form ω. The case that, for a fixed choice of 1-isomorphisms and 2-isomorphisms, a different Z-simplicial open cover has been used can be reduced to the case that one cover refines the other. Then, local data for the finer cover can be chosen as the restriction of the local data for the coarser one. In this case, the identification (8.2) produces the same U(1)-numbers and hence the same cocycle u. Presume further that we either had chosen a different multiplicative or equivariant structure on $\mathcal{G}$, or chosen different local data. Due to the 2-connectedness of G one can then find local data (r, s, t) of an equivalence between Z-multiplicative gerbes, relating local data $(c, b, a, \beta, \phi, d)$ to other local data $(c', b', a', \beta', \phi', d')$ by

$$c_2 = c_1 + D_{1,0,1}r \quad , \quad b_2 = b_1 + \delta_{1,0,1}r + D_{1,1,0}s \quad \text{and} \quad \beta_2 = \beta_1 + \Delta_{1,0,1}r + D_{2,0,0}t.$$

These are eqs. (7.10) and (7.11) minus the equations corresponding to the commutativity of diagrams of 2-isomorphisms, which are not automatically guaranteed. Anyway, it is now easy to see that

$$x := a' - a + \delta_{1,1}s \quad , \quad y := \phi' - \phi - \Delta_{1,1}s + \delta_{2,0}t \quad \text{and} \quad z := d' - d + \Delta_{2,0}t$$

defines a 3-cochain (x, y, z) in the total complex of $C^q(Z^p, \mathrm{U}(1))$, whose coboundary is $u' - u$. Thus, the class $[u]$ is well-defined.

The second claim about the class $[u]$ is that it is the obstruction against the existence of a Z-multiplicative structure for the pair $(\mathcal{G}, \omega)$, i.e. there exists a Z-multiplicative structure if and only if $[u] = 0$. The "only if" part is trivial: if there is a Z-multiplicative structure on $(\mathcal{G}, \omega)$, the corresponding cocycle u is identically zero, since all the relations (8.1) are satisfied. Conversely, suppose (x, y, z) is a 3-cochain whose coboundary is $(u_0, \ldots, u_4)$. Then, the new local data $(c, b, a - x, \rho, \beta, \phi - y, d - z)$ for a Z-multiplicative gerbe satisfies all required conditions. Reconstructing the gerbe, isomorphisms and 2-isomorphisms from this local data yields a Z-multiplicative gerbe with the underlying gerbe isomorphic to $\mathcal{G}$ and with the 2-form ω. The latter isomorphism allows to carry the Z-multiplicative structure to $\mathcal{G}$.

Now we analyze the obstruction class $[u] \in \mathbb{H}^4_0(Z, \mathrm{U}(1))$ in detail. As the cocycle conditions (8.4) show, the obstructions split into one obstruction $u_3 \in \mathrm{U}(1)$ and a class $[(u_0, u_1, u_2)]$ in $\mathbb{H}^4(Z, \mathrm{U}(1))$, the total cohomology of the double complex $C^q(Z^p, \mathrm{U}(1))$ with the $(q = 0)$-row omitted. Since the differential $\Delta_{3,0}$ is the identity, we can always find a cohomologous 4-cocycle with $u_3 = 0$. And because the differential $\delta_{3,0}$ is the zero map, the class $[(u_0, u_1, u_2)] \in \mathbb{H}^4(Z, \mathrm{U}(1))$ is trivial if and only if the class $[u] \in \mathbb{H}^4_0(Z, \mathrm{U}(1))$ is trivial. Summarizing, there is one well-defined obstruction $[(u_0, u_1, u_2)] \in \mathbb{H}^4(Z, \mathrm{U}(1))$ against a Z-multiplicative structure on $(\mathcal{G}, \omega)$.

The obstruction class $[u]$ can further be treated as follows. Since $\delta_{1,3}u_0 = 0$, we have a class $[u_0] \in H^3(Z,\mathrm{U}(1))$. Suppose this class is trivial and that $x \in C^2(Z,\mathrm{U}(1))$ is such that $\delta_{1,2}x = u_0$. If we now change u by the coboundary of $(x,0)$, we obtain a new representative $u^x := (0, u_1^x, u_2)$ with $u_1^x := u_1 - \Delta_{1,2}x$, in which $\delta_{2,2}u_1^x = 0$. It defines a class $[u_1^x] \in H^2(Z^2,\mathrm{U}(1))$. Note that this class depends on the choice of x. Now suppose that the class $[u_1^x]$ is trivial, i.e. there exists $y \in C^1(Z^2,\mathrm{U}(1))$ such that $\delta_{2,1}y = u_1^x$. We change u^x be the coboundary of $(0,y)$ and obtain a new representative $u^{x,y} = (0,0,u_2^y)$ with $u_2^y := u_2 - \Delta_{2,1}y$. The cocycle conditions are now $\delta_{3,1}u_2^y = 0$ and $\Delta_{3,1}u_2^y = 0$. This relations means that u_2^y is a character on Z^3 that is also a 3-cocycle if viewed as a 3-chain on Z. A short computation shows that $(u_2^y)_{(z,z',z'')} = \chi(z) + \chi'(z'')$, where χ and χ' are characters on Z. Upon setting $v_(z,z') = -\chi(z) + \chi'(z')$, one checks that $v \in C^1(Z^2,\mathrm{U}(1))$ satisfies $\Delta_{2,1}v = u_2^y$ and $\delta_{2,1}v = 0$. Altogether, this implies that the cocycle $u = (u_0,u_1,u_2)$ is a coboundary of $(x, y+v)$. Thus, the obstruction $[u] \in \mathbb{H}^4(Z,\mathrm{U}(1))$ is trivial if, and only if, successively, the class of u_0 in $H^3(Z,\mathrm{U}(1))$ and the class of u_1^x in $H^2(Z^2,\mathrm{U}(1))$ vanish.

Tracing back through the extraction of local data, we see that u_0 is the error in the commutativity of the diagram (6.7) for the 2-isomorphism φ_{z_1,z_2} of equivariant structures. Thus, its class $[u_0] \in H^3(Z,\mathrm{U}(1))$ is the well-known obstruction from [22, 26] to the existence of the descent gerbe $\mathcal{G}'$ on $G' = G/Z$. Further, once an equivariant structure (the local datum x) is chosen, we see that u_1^x is the error in the commutativity of the diagram which is needed to make $(\mathcal{M}, \eta_{z_1,z_2})$ a Z^2-equivariant isomorphism. Thus, the class $[u_1^x] \in H^2(Z^2,\mathrm{U}(1))$ obstructs the existence of the descent isomorphism $\mathcal{M}'$ in the multiplicative structure on the gerbe $\mathcal{G}'$.

For the particular case of $\mathcal{G}_k$ the basic gerbe over a compact, simple and simply-connected Lie group G, and ω_k the 2-form (1.5), we have shown in section 5, that the class $\kappa = [u_1^x]$ is trivial if and only if the FGK cocycle c associated to $\mathcal{G}'_k$ is trivial (note that the groups G and $G' = G/Z$ play here the role of $\tilde{G}$ and $G = \tilde{G}/Z$ from the first sections of the paper where the discussion was centered on the gerbes over non-simply connected groups). Thus, the calculation of the FGK cocycle carried out in section 4 identifies precisely the the situations for which $\mathcal{G}'_k$ is a multiplicative gerbe.

9 Uniqueness of multiplicative structures

In this section we address the question if there are inequivalent choices of Z-multiplicative structures on a pair $(\mathcal{G},\omega)$ of a gerbe $\mathcal{G}$ over a 2-connected Lie group G and a compatible 2-form ω. First we claim that equivalence classes of Z-multiplicative structures on $(\mathcal{G},\omega)$, if they exist, are parameterized by $\mathbb{H}^3(Z,\mathrm{U}(1))$.

Let us first see how $\mathbb{H}^3(Z,\mathrm{U}(1))$ acts on equivalence classes of Z-multiplicative gerbes. A 3-cocycle consists of cochains $x \in C^2(Z,\mathrm{U}(1))$ and $y \in C^1(Z^2,\mathrm{U}(1))$ such that

$$\delta_{1,2}x = 0 \quad , \quad \Delta_{1,2}x + \delta_{2,1}y = 0 \quad \text{and} \quad \Delta_{2,1}y = 0.$$

If (c,b,a,β,ϕ,d) is local data for a Z-multiplicative gerbe then $(c,b,a+x,\beta,\phi+y,d)$ is local data for another one. Suppose that (x',y') is a cohomologous cocycle, i.e. there exists $s \in C^1(Z,\mathrm{U}(1))$ such that $x' = x + \delta_{1,1}s$ and $y' = y + \Delta_{1,1}s$. Then, $(0,s,0)$ is local data

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for an equivalence between $(c,b,a+x,\beta,\phi+y,d)$ and $(c,b,a+x',\beta,\phi+y',d)$. Thus, the action of $\mathbb{H}^3(Z,\mathrm{U}(1))$ on equivalence classes of Z-multiplicative gerbes is well-defined. It is also free: whenever (c,b,a,β,ϕ,d) and $(c,b,a+x,\beta,\phi+y,d)$ are equivalent local data for an equivalence (r,s,t), one can choose $r=t=0$ and verify that the coboundary of s is (x,y). It remains to show that the action is transitive.

If (c,b,a,β,ϕ,d) and $(c',b',a',\beta',\phi',d')$ are local data for Z-multiplicative gerbes over G, with c and c' local data for the fixed gerbe $\mathcal{G}$, one can always find local data (r,s,t) of an equivalence that Change the second to $(c,b,a'',\beta,\phi'',d'')$, as mentioned in section 8. One can also arrange t such that $d''=d$, due to the fact that multiplicative structures on $\mathcal{G}$ are unique [18]. Defining $x:=a''-a$ and $y:=\phi''-\phi$, we obtain a class $[(x,y)]\in\mathbb{H}^3(Z,\mathrm{U}(1))$ whose action connects the two sets of local data. Thus, the action of $\mathbb{H}^3(Z,\mathrm{U}(1))$ is transitive.

We have shown so far that $\mathbb{H}^3(Z,\mathrm{U}(1))$ parameterizes inequivalent Z-multiplicative structures on a gerbe $\mathcal{G}$ over a 2-connected Lie group G. Now we calculate $\mathbb{H}^3(Z,\mathrm{U}(1))$ for the groups $Z=\mathbb{Z}_N$ and $Z=\mathbb{Z}_2^2$ that appear as subgroups of centers of compact, simple and simply-connected Lie groups. We first discuss the case of $Z=\mathbb{Z}_N$, for which we consider a 3-cocycle (x,y). Since $H^2(\mathbb{Z}_N,\mathrm{U}(1))=0$ according to (2.9), we can go to a representative with $x=0$. Then, $\delta_{2,1}y=0$ and $\Delta_{2,1}y=0$, i.e. $y\in C^1(Z^2,\mathrm{U}(1))$ is a character on Z^2 that is also a 2-cocycle when viewed as a 2-chain on Z. A simple calculation shows that $y=0$. Thus, $\mathbb{H}^3(\mathbb{Z}_N,\mathrm{U}(1))=0$. We continue with $Z=\mathbb{Z}_2^2$. Here, $H^2(Z,\mathrm{U}(1))=\mathbb{Z}_2$, see (2.9). One can represent the non-trivial class explicitly. Under the embedding (5.6),

$$\mathrm{Hom}(\mathbb{Z}_2,\mathrm{U}(1))\cong\mathrm{Hom}(\mathbb{Z}_2\otimes\mathbb{Z}_2,\mathrm{U}(1))\hookrightarrow H^2(\mathbb{Z}_2^2,\mathrm{U}(1)),$$

the non-trivial group homomorphism $\zeta:\mathbb{Z}_2\to\mathrm{U}(1)$ maps to the non-trivial class represented by $\chi_{(z_1,z_2),(z_1',z_2')}:=\zeta(z_1z_2')$. We can thus check explicitly that

$$(\Delta_{1,2}\chi)_{((z_1,z_2),(\tilde{z}_1,\tilde{z}_2)),((z_1',z_2'),(\tilde{z}_1',\tilde{z}_2'))}=0.$$

It follows as before that $y=0$. Consequently, $\mathbb{H}^3(\mathbb{Z}_2^2,\mathrm{U}(1))=\mathbb{Z}_2$. This is just the well-known choice of the $\mathbb{Z}_2^2$-equivariant structure on the gerbe $\mathcal{G}_k$ over $\mathrm{Spin}(4r)$ [22, 26].

Summarizing, once an equivariant structure on the gerbe $\mathcal{G}_k$ over G is fixed, the multiplicative structure on the descent gerbe $\mathcal{G}_k'$ is, if it exists, unique up to isomorphism.

10 Conclusions

We have studied obstructions to the existence of multiplicative structures on (bundle) gerbes $\mathcal{G}_k$ over simple compact groups G, i.e. on gerbes with connection of curvature H_k given by eq. (1.1). This was done by analyzing the multiplicative gerbes over the universal covering groups $\tilde{G}$ equivariant w.r.t. the deck action of the fundamental group $\pi_1(G)=Z$. We have shown that there are two obstructions to the existence of such equivariant multiplicative gerbes over $\tilde{G}$. The first one lies in the cohomology groups $H^3(Z,\mathrm{U}(1))$. Its triviality assures the existence of the gerbe $\mathcal{G}_k$ over the quotient group $G=\tilde{G}/Z$. Such a gerbe determines unambiguously the Feynman amplitudes in the group G WZW theory over closed oriented surfaces. Given the gerbe $\mathcal{G}_k$ over G, the second obstruction lies in

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$\tilde{G}$	Center	Z	WZW constraints on k	CS constraints on k
$\mathrm{SU}(r)$	$\mathbb{Z}_r$	$Z = \mathbb{Z}_N$ with $N \mid n$	$2N \mid kr(r-1)$	$2N^2 \mid kr(r-1)$
$\mathrm{Spin}(2r+1)$	$\mathbb{Z}_2$	$Z = \mathbb{Z}_2$	–	$2 \mid k$
$\mathrm{Spin}(4r+2)$	$\mathbb{Z}_4$	$Z = \mathbb{Z}_2$	–	$2 \mid k$
		$Z = \mathbb{Z}_4$	$2 \mid k$	$8 \mid k$
$\mathrm{Spin}(4r)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$Z = \mathbb{Z}_2 \times \{0\}$	$2 \mid kr$	$4 \mid kr$
		$Z = \{0\} \times \mathbb{Z}_2$	–	$2 \mid k$
		$Z = \{(0,0),(1,1)\}$	$2 \mid kr$	$4 \mid kr$
		$Z = \mathbb{Z}_2 \times \mathbb{Z}_2$	$2 \mid kr$	$2 \mid k$ and $4 \mid kr$
$\mathrm{Sp}(2r)$	$\mathbb{Z}_2$	$Z = \mathbb{Z}_2$	$2 \mid kr$	$4 \mid kr$
E_6	$\mathbb{Z}_3$	$Z = \mathbb{Z}_3$	–	$3 \mid k$
E_7	$\mathbb{Z}_2$	$Z = \mathbb{Z}_2$	$2 \mid k$	$4 \mid k$

Table 1. Constraints for integer values of k imposed by the consistency of, respectively, the WZW and the CS theory with group $G = \tilde{G}/Z$.

the cohomology group $H^2(Z^2, \mathrm{U}(1))$. Its triviality, equivalent to the triviality of the FGK 2-cocycle on the group Z^2, guarantees the existence of the multiplicative structure on the gerbe $\mathcal{G}_k$. The gerbe $\mathcal{G}_k$ with such a structure determines unambiguously the Feynman amplitudes in the group G CS theory over closed oriented 3-dimensional manifolds. We made explicit the relation between the obstruction cohomology class in $H^2(Z^2, \mathrm{U}(1))$ and the FGK cocycle by relating both to Kreuzer-Schellekens bihomomorphisms. The constraints on the integer levels k imposed by the triviality of, respectively, the first obstruction or the first and the second one are collected in the table in figure 2. Over the groups $G = \mathrm{Spin}(4r)/\mathbb{Z}_2^2$ where there are two inequivalent gerbes $\mathcal{G}_k$ when kr is even, only one of them admits multiplicative structure when kr is divisible by 4. For simplicity, we have limited our geometric considerations to the case of simple compact groups. The extension of our analysis to the case of non-simple compact groups, treated within the simple-current orbifold algebraic approach in [9–12, 21] does not seem to present difficulties.

Multiplicative structures on gerbes over groups may be viewed as conditions of compatible equivariance of gerbes under the group actions on itself by the left and the right multiplications. We shall develop elsewhere a theory of gerbes equivariant under actions of continuous groups. Such gerbes permit the treatment of sigma models with the gauged Wess-Zumino actions, e.g. the coset models of conformal field theory [30]. One of the useful applications of the multiplicative structures on the gerbe $\mathcal{G}_k$ over group G is that such a structure induces an equivariant structure w.r.t. the adjoint action of G on itself (although the latter exists also when there is no multiplicative structure on $\mathcal{G}_k$). A multiplicative structure on gerbes $\mathcal{G}_k$ induces also two Jandl structures on $\mathcal{G}_k$ (by the pullback of 1-isomorphisms $\mathcal{M}$ of (6.4) from G^2 to G via the maps $g \mapsto (g, g^{-1})$ or $g \mapsto (g^{-1}, g)$. Such structures are used to define the amplitudes of the WZW theory over unoriented surfaces [26, 31]. Finally, multiplicative structure on gerbes $\mathcal{G}_k$ play an important role in WZW theory with defects permitting to define symmetric bi-branes [18, 32]. A detailed study of such defects for non-simply connected groups is another problem left for future research.

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A Simplicial and equivariant refinements of sequences of open covers

Let G be a Lie group, and let Z be a finite Abelian group with a smooth, free, proper and multiplicative action on G. Suppose that $\{\mathfrak{V}^p\}$ is a sequence consisting of open covers $\mathfrak{V}^\mathfrak{p}$ of G^p for $p > 0$. In this appendix we show that there exists a Z-simplicial sequence $\{\mathfrak{O}^p\}$ open covers in the sense of section 7, such that each $\mathfrak{O}^p$ is a refinement of $\mathfrak{V}^p$.

We have to introduce some simplicial techniques. Let $\mathcal{P}$ be the category whose objects are non-negative integer numbers $0, 1, 2, \ldots$ and whose set $\mathcal{P}(n, m)$ of morphisms from n to m consists of maps $\theta : \{0, \ldots, n\} \to \{0, \ldots, m\}$ such that $i < j$ implies $\theta(i) < \theta(j)$. If $n > m$, the set $\mathcal{P}(n, m)$ is empty. If $n = m$, it contains only the identity. Furthermore, for any $n > 0$ the set $\mathcal{P}(n - 1, n)$ consists of $(n + 1)$ elements, the "universal face maps" θ_k, where $0 \leq k \leq n$, defined by $\theta_k(i) := i$ for $i < k$ and $\theta_k(i) := i + 1$ for $i \geq k$. The category $\mathcal{P}$ has the following well-known purpose. There is a bijection between incomplete simplicial sets and contravariant functors $X : \mathcal{P} \to \mathcal{S}et$, where $\mathcal{S}et$ denotes the category of sets. Given such a functor, one obtains an incomplete simplicial set $\{X^p\}$ by setting $X^p := X(p)$ and $\Delta^p_k := X_{\theta_k} := X(\theta_k)$ for $\theta_k \in \mathcal{P}(p - 1, p)$ one of the universal face maps. Conversely, one can write any $\theta \in \mathcal{P}(n, m)$ as a composition of universal face maps and then invert this construction. In particular, every group G defines a contravariant functor $G : \mathcal{P} \to \mathcal{S}et$, corresponding to the incomplete simplicial set $\{G^p\}$ that we considered at the beginning of section 6. An analogous statement is true in the category $\mathcal{M}an$ of smooth manifolds: the contravariant functors $M : \mathcal{P} \to \mathcal{M}an$ are in bijection to incomplete simplicial manifolds.

First we recall a general construction of [33]. Let $M : \mathcal{P} \to \mathcal{M}an$ be a contravariant functor and let $\{\mathfrak{V}^p\}$ be a sequence of open covers, with $\mathfrak{V}^p = \{V^p_j\}_{j\in J^p}$ an open cover of the manifold M^p. In the following we use the notation

$$\mathcal{P}^p := \bigcup_{k=0}^{p} \mathcal{P}(k, p) \quad \text{and} \quad \mathcal{J}^p := \bigcup_{k=0}^{p} J^k.$$

A new open cover $\mathfrak{O}^p$ of M^p is defined as follows. Its index set is

$$I^p := \Big\{ i : \mathcal{P}^p \to \mathcal{J}^p \ \big| \ i(\theta) \in J^k \text{ for } \theta \in \mathcal{P}(k, p) \Big\} .$$

Its open sets are

$$O^p_i := \bigcap_{k=0}^{p} \ \bigcap_{\theta\in\mathcal{P}(k,p)} M^{-1}_\theta \left(V^k_{i(\theta)} \right) , \tag{A.1}$$

where $M_\theta : M^p \to M^k$ is the smooth map assigned to θ. These open sets cover M^p: for $x \in M^p$ choose an index $j_\theta \in J^k$ for each $\theta \in \mathcal{P}(k, p)$ such that $M_\theta(x) \in V^k_{j_\theta}$. The assignment $\theta \mapsto j_\theta$ defines an index $i \in I^p$, and it is clear that $x \in O^p_i$. Furthermore, $\mathfrak{O}^p$ is a refinement of $\mathfrak{V}^p$: with $r : I^p \to J^p$ defined by $r(i) := i(\mathrm{id}_p)$ we have immediately $O^p_i \subset V^p_{r(i)}$.

Next we define a contravariant functor $I : \mathcal{P} \to \mathcal{S}et$ with $I(p) := I^p$, turning the sequence $\{I^p\}$ of index sets into an incomplete simplicial set. For $\phi \in \mathcal{P}(n, m)$, we let $I_\phi : I^m \to I^n$ be defined by

$$I_\phi(i)(\theta) := i(\phi \circ \theta) \in J^k$$

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for $i \in I^m$ and $\theta \in \mathcal{P}(k,n)$. It is clear that this definition yields a functor. Now we have all the structure of a simplicial sequence of open covers, and it remains to check condition (7.1). In fact the more general relation

$$M_\theta(O_i^p) \subset O^\ell_{I_\theta(i)} \tag{A.2}$$

is true for all $i \in I^p$ and $\theta \in \mathcal{P}(\ell,p)$, from which (7.1) follows by restricting to the universal face maps $\theta_k \in \mathcal{P}(p-1,p)$. To show (A.2) we have to prove that $M_\theta(O_i^p)$ is contained in all the open sets $M_\phi^{-1}(V^k_{I_\theta(i)(\phi)})$ that form the intersection (A.1). Indeed,

$$M_\phi\left(M_\theta(O_i^p)\right) = M_{\theta\circ\phi}(O_i^p) \subset \bigcap_{k=0}^{p} \bigcap_{\psi\in\mathcal{P}(k,p)} M_{\theta\circ\phi}\left(M_\psi^{-1}(V^k_{i(\psi)})\right) \subset V^k_{i(\theta\circ\phi)} = V^k_{I_\theta(i)(\phi)}.$$

Here, the last inclusion follows by restricting to $\psi = \theta\circ\phi$. Summarizing the construction we took from [33], the sequence $\{\mathfrak{O}^p\}$ of open covers is simplicial, and each cover $\mathfrak{O}^p$ is a refinement of the original open cover $\mathfrak{V}^p$.

Now we enhance the construction above by additional Z-equivariance. Since we have a free and proper group action of a finite group it is clear that each open cover $\mathfrak{V}^p$ has a Z^p-equivariant refinement. We can thus assume that such refinements are already chosen, and that the sequence $\{\mathfrak{V}^p\}$ is Z-equivariant. It remains to prove that applying the above construction to a Z-equivariant sequence $\{\mathfrak{V}^p\}$ yields a Z-simplicial sequence. To start with, the action of Z^p on the index set I^p is defined by

$$(z.i)(\theta) := Z_\theta(z).i(\theta), \tag{A.3}$$

for $\theta \in \mathcal{P}(k,p)$ and $z \in Z^p$. Here $Z_\theta : Z^p \to Z^k$ is the map the functor $Z : \mathcal{P} \to \mathcal{S}et$ associated to the group Z assigns to θ, and on the right hand side we have used the action of $Z_\theta(z) \in Z^k$ on the index $i(\theta) \in J^k$ of the Z^k-equivariant open cover $\mathfrak{V}^p$. Definition (A.3) yields an action because Z_θ is a group homomorphism; here we use that Z is Abelian. Notice that the refinement maps $r : I^p \to J^p$ defined above are Z^p-equivariant. Next we prove the relation (7.4) for Z-equivariant covers: first we have

$$z(O_i^p) \subset \bigcap_{k=0}^{p} \bigcap_{\theta\in\mathcal{P}(k,p)} z\left(M_\theta^{-1}(V^k_{i(\theta)})\right) \subset \bigcap_{k=0}^{p} \bigcap_{\theta\in\mathcal{P}(k,p)} M_\theta^{-1}\left(Z_\theta(z)(V^k_{i(\theta)})\right),$$

for $z \in Z^k$. Here we have used that the action is multiplicative; more specifically that diagram (6.12) is commutative. Then it remains to check that

$$M_\theta^{-1}\left(Z_\theta(z)(V^k_{i(\theta)})\right) \subset M_\theta^{-1}\left(V^k_{Z_\theta(z).i(\theta)}\right) = M_\theta^{-1}\left(V^k_{(z.i)(\theta)}\right).$$

This shows that the sequence $\{\mathfrak{O}^p\}$ of open covers is Z-equivariant. It remains to check the compatibility condition (7.6) between the face maps of $\{I^p\}$ and the actions of Z^p on I^p. Indeed, for $\theta \in \mathcal{P}(p-1,p)$, $\phi \in \mathcal{P}(k,p-1)$, $z \in Z^p$ and $i \in I^p$ we find

$$(I_\theta(z.i))(\phi) = (z.i)(\theta\circ\phi) = Z_{\theta\circ\phi}(z).i(\theta\circ\phi) = Z_\phi(Z_\theta(z)).(I_\theta(i)(\phi)) = (Z_\theta(z).I_\theta(i))(\phi),$$

showing the commutativity of (7.6). Summarizing, $\{\mathfrak{O}^p\}$ is a Z-simplicial sequence of open covers.

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